Concepts:

- Evaluate values of recursively defined functions.
- Find members of recursively defined sets and arrange them in a tree diagram which shows the parent-child relationship.
- State the "Principle of Structural Induction."
- Prove statements about recursively defined sets using structural induction.
- Identify common mistakes in incorrect proofs.

Problems:

1. Explain similarities and differences between structural induction proofs on a recursively defined set and linear induction proofs on the set of positive integers.

2. The set S is recursively defined by:

   - $1 \in S$
   - $x \in S \rightarrow 5x - 1 \in S$
   - $x \in S \rightarrow 7x \in S$.

   Suppose you want to use structural induction to show that all members of S are in the form $3k + 1$ for some nonnegative integer k. Choose the correct option for the following steps in your proof.

   (a) What do you have to prove in the Basis Step?

      i. That $1 \in S$ and 1 is in the form $3k + 1$ for some nonnegative integer k.

      ii. That 1 is in the form $3k + 1$ for some nonnegative integer k.

      iii. That the first generation of elements 4 and 7 are in the form $3k + 1$ for some nonnegative integer k.

      iv. That the first generation of elements 4 and 7 are in S, and they are in the form $3k + 1$ for some nonnegative integer k.

   (b) What do you have to prove in the Recursive Step?

      i. That the first generation of elements 4 and 7 are in the form 3k+1 for some nonnegative integer k.

      ii. That if x is an arbitrary element of S and $5x - 1, 7x$ are in the form $3k + 1$ for some nonnegative integer k, then so is x.

      iii. That if x is an arbitrary element of S, then $5x - 1, 7x$ are also elements of S.

      iv. That if x is an arbitrary element of S in the form $3k + 1$ for some nonnegative integer k, so are $15x - 1$ and $7x$.

3. Give a recursive definition for the following sets.

(a) The set of positive even integers.

(b) The set of negative odd integers.

(c) The set of positive integers that are not divisible by 4.

(d) The set of integers that are of the form of $2 \cdot 3^k$ for some nonnegative integer k.

(e) The set of ordered pairs (a, b) such that a, b are non-negative integers and $a - b$ is odd.

4. Find the mistake(s) in the following proofs:

Suppose that the set S is recursively defined by:

- $2 \in S$
- $x \in S \to 5x + 1 \in S$.

Prove by structural induction that each element in S is of the form $3k + 2$ for some nonnegative integer k.

(a) **Proof:**

Basis Step The initial population 2 has the form of $3k+2$ for some nonnegative integer k, since $2 = 3\cdot0+2$.

Recursive Step: Assume x is an arbitrary element in S. Then $x = 3k+2$ for some nonnegative integer k.

Applying the recursive definition to x, we obtain $5x + 1 = 5(3k + 2) + 1 = 15k + 11$.

We have proved that the direct descendant of x is in the form $3k + 2$ for some nonnegative integer k.

Thus, by the Principle of Structural Induction, all elements of S are in the form $3k + 2$ for some nonnegative integer k.

(b) **Proof:**

Basis Step: The initial population 2 has the form of $3k + 2$ for some nonnegative integer k, since $2 = 3 \cdot 0 + 2$. If 2 is in S, then 11 is in S. $11 = 3 \cdot 3 + 2$, therefore 11 can be written in the form $3k + 2$ for $k = 3$ which shows that 11 has the required form.

Recursive Step: Assume x is an arbitrary element in S such that $x = 3k + 2$ for some nonnegative integer k.

Then $5x + 1 = 5(3k + 2) + 1 = 15k + 11 = 3 \cdot (5k + 3) + 2$.

Since $5k + 3$ is an integer, we have showed that the direct descendant of x is in the form $3k + 2$ for some nonnegative integer k.

Thus, by the Principle of Structural Induction, all elements of S are in the form $3k + 2$ for some nonnegative integer k.

(c) **Proof:**

Basis Step: The initial population 2 has the form of $3k+2$ for some nonnegative integer k, since $2 = 3 \cdot 0 + 2$.

Recursive Step: Assume x is an arbitrary element in S such that $x = 3k+2$ for some nonnegative integer k.

If 2 is in S, then $5 \cdot 2 + 1 = 11$ is in S and $11 = 3 \cdot 3 + 2$.

If 11 is in S, then $5 \cdot 11 + 1 = 56$ is in S and $56 = 3 \cdot 14 + 2$ and so on.

We see that all descendants are in the form $3k+2$ for some nonnegative integer k, and hence the statement is proved by the Principle of Structural Induction.

(d) **Proof:**

Basis Step: The initial population 2 has the form of $3k+2$ for some nonnegative integer k, since $2 = 3 \cdot 0 + 2$.

Recursive Step: Assume x is an arbitrary element in S such that $x = 3k+2$ for some nonnegative integer k. We need to show that the descendant $5x+1$ is also in S and has the required form.

Then $5x + 1 = 5(3k+2) + 1 = 15k + 11 = 3 \cdot (5k+3) + 2$.

Since $5k+3$ is a nonnegative integer, we have proved that $5x+1$, the descendant of x, is in S and has the required form.

Thus, by the Principle of Structural Induction, we have proved that all elements of S are in the form $3k+2$ for some nonnegative integer k.

(e) **Proof:**

Let $P(x)$ denote the statement that $x = 3k+2$ for some non-integer k when x is in S.

Basis Step: $P(2)$ is true since $2 = 3 \cdot 0 + 2$ which shows that the initial population has the required form.

Recursive Step: Assume $P(x)$ is true for an arbitrary element x in S. We need to show $P(x+1)$, that is, the descendant $5(x+1)+1$ can also be written in the form $3k+2$ for some nonnegative integer k.

$5(x+1) + 1 = 5x + 6 = 3(\frac{5}{3}x + \frac{4}{3}) + 2$ which proves $P(x+1)$.

Thus, by the Principle of Structural induction, the statement $P(x)$ is proved for all elements x in S.

5. Find the mistake(s) in the following proof:

Assume S is a recursively defined set of ordered pairs defined by the following properties:

- $(1,3) \in S$
- $(a,b) \in S \to (a+1, b+3) \in S$

Use structural induction to prove that for all members (a,b) of S, $3a \leq b$.

Proof:

Basis Step: The initial population $(1,3)$ satisfies the property since $3 \cdot 1 \leq 3$.

Recursive Step: Assume that (a,b) is an arbitrary member of S such that $3 \cdot a \leq b$. By the recursive hypothesis

$3(a+1) \leq b+3 = 3a \leq b$, which implies the descendant $(a+1, b+3)$ has the required property.

Thus, $3 \cdot a \leq b$ for all elements (a,b) of S by the Principle of Structural Induction.

6. Find the mistake(s) in the following proof:

   Assume S is a recursively defined set of ordered pairs defined by the following properties:

   - $(1,-5) \in S$
   - $(a,b) \in S \to (a+1, b+2) \in S$

   Use structural induction to prove that for all members (a,b) of S, $3a+2b$ is a multiple of 7.

   Proof:

   Basis Step: The initial population $(1,-5)$ satisfies the property since $3 \cdot 1 + 2 \cdot (-5) = -7$ and -7 is a multiple of 7

   Recursive Step: Assume that (a,b) is an arbitrary member of S such that $3a+2b$ is a multiple of 7, that is, $3a+2b = 7k$ for some integer k. By the recursive hypothesis

   $(a+1, b+2) = 3(a+1)+2(b+2) = (3a+2b)+7 = 7k+7 = 7(k+1)$, which implies the descendant $(a+1, b+2)$ is a multiple of 7.

   Thus, $3a+2b$ is a multiple of 7 for all elements (a,b) of S by the Principle of Mathematical Induction.

7. Suppose the set S is recursively defined by:

   - $1 \in S$
   - $x \in S \to 5x-1 \in S$
   - $x \in S \to 7x \in S$
   - $x \in S \to x^2 \in S$.

   Prove by structural induction that $x \bmod 3 = 1$ for all $x \in S$.

8. Suppose the set S is recursively defined by:

   - $2 \in S$
   - $x \in S \to 2x+1 \in S$
   - $x \in S \to x^2+1 \in S$.

   Prove by structural induction that $x \bmod 3 = 2$ for all $x \in S$.

9. Suppose the set S is recursively defined by:

- $2 \in S$
- $x \in S \rightarrow x^2 \in S$
- $x \in S \wedge y \in S \rightarrow xy^3 \in S$.

Prove by structural induction that each member of S is a power of 2, i.e., it is equal to 2^k for some positive integer k.

10. Suppose the set S is recursively defined by:
    - $2 \in S$
    - $3 \in S$
    - $x \in S \rightarrow x^{-2} \in S$
    - $x \in S \wedge y \in S \rightarrow xy \in S$.

    Prove by structural induction that each member of S is of the form $2^k 3^l$ for some integers k and ℓ.

11. Suppose the set S is recursively defined by:
    - $(0, 3) \in S$
    - $(x, y) \in S \rightarrow (x + 2, y - 1) \in S$
    - $(x, y) \in S \rightarrow (x - 3, y) \in S$

    Prove by structural induction that $x - y \bmod 3 = 0$ for all $(x, y) \in S$.

12. Suppose the set S is recursively defined by:
    - $(0, 1) \in S$
    - $(x, y) \in S \rightarrow (x + 1, y - 1) \in S$
    - $(x, y) \in S \rightarrow (x + 2, y) \in S$

    Prove by structural induction that $x - y$ is odd for all $(x, y) \in S$.

13. Suppose B is the set of bit strings recursively defined by:
    - $010 \in B$
    - $b \in B \rightarrow 00b \in B$
    - $b \in B \rightarrow b1 \in B$

    Prove by structural induction that all members of B contain an even number of zeros.

14. Suppose B is the set of bit strings recursively defined by:
    - $01 \in B$
    - $b \in B \rightarrow 0b \in B$
    - $b \in B \rightarrow b1 \in B$

    Prove by structural induction that B does not contain any string in which a 0 occurs to the right of a 1.

Solutions:

1. Explain similarities and differences between structural induction proofs on a recursively defined set and linear induction proofs on the set of positive integers.

   Linear induction is just a special case of a structural induction, where the recursion rule is the following:

   - $1 \in \mathbb{N}$
   - $n \in \mathbb{N} \rightarrow n+1 \in \mathbb{N}$

   and we prove that the property $P(n)$ is true for all $n \in \mathbb{N}$. Thus, linear induction on the set of positive integers is a structural induction with a linear recursion tree.

   In the Basis Step of a linear induction proof, we prove that the statement $P(n)$ is true for $n = 1$, that is, we prove $P(1)$.

   In the Basis Step of a structural induction proof, we prove that a certain property P is true for all elements x in the initial population S_0, that is, we prove $P(x)$ for all $x \in S_0$.

   In the Inductive Step of a linear induction proof, we prove that if the statement P is true for a positive integer n, then the statement P is also true for $n + 1$. In other words, we prove the conditional statement $P(n) \rightarrow P(n + 1)$ for all positive integers n.

   In the Recursive Step of a structural induction proof, we prove that if the property P is true for an element x of S, then P is also true for each direct descendant of x. Formally, we prove the conditional statement $P(x) \rightarrow P(f_k(x))$ for all $x \in S$ and for all recursive rules f_k defined on S.

   While in a linear induction proof, the logical chain goes from n to $n + 1$, in a structural induction proof the recursive step follows the tree structure of the recursively defined set, which is not necessarily linear.

2. The set S is recursively defined by:

   - $1 \in S$
   - $x \in S \rightarrow 5x - 1 \in S$
   - $x \in S \rightarrow 7x \in S$.

   Suppose you want to use structural induction to show that all members of S are in the form $3k + 1$ for some nonnegative integer k. Choose the correct option for the following steps in your proof.

   (a) What do you have to prove in the Basis Step?

      i. That $1 \in S$ and 1 is in the form $3k + 1$ for some nonnegative integer k.

         $1 \in S$ by the definition of the set. We do not need to prove that.

      ii. That 1 is in the form $3k + 1$ for some nonnegative integer k.

         Correct.

      iii. That the first generation of elements 4 and 7 are in the form $3k + 1$ for some nonnegative integer k.

         Proving that the first generation of elements are in the form $3k + 1$ does not belong to the Basis Step. In fact, it does not belong anywhere in the proof. The Recursive Step takes care of it.

iv. That the first generation of elements 4 and 7 are in S, and they are in the form $3k+1$ for some nonnegative integer k.

4 and 7 are in S according to the definition of S. We do not prove memberships that are given by the defining rules.

The Recursive Step includes the proof of the property for first generation of elements, so we do not prove it in the Basis Step.

(b) What do you have to prove in the Recursive Step?

i. That the first generation of elements 4 and 7 are in the form $3k+1$ for some nonnegative integer k.

Proving that the first generation of elements have the property is not part of the Recursive Step. In fact, it does not belong anywhere in the proof. The Recursive Step takes care of proving this property if it is properly set up.

ii. That if x is an arbitrary element of S and $5x-1$ and $7x$ are in the form $3k+1$ for some nonnegative integer k, then so is x.

This approach assumes the conclusion. It assumes that the children $5x-1$ and $7x$ have the property, and requires proving that the parent x also has this property. The logic should be reversed and the converse of the statement should be proved. The property inherits from the parent to the child.

iii. That if x is an arbitrary element of S, then $5x-1$ and $7x$ are also elements of S.

$5x-1$ and $7x$ are in S according to the definition of S. We do not prove memberships that are given by the rules.

iv. That if x is an arbitrary element of S in the form $3k+1$ for some nonnegative integer k, so are $15x-1$ and $7x$.

Correct.

3. Give a recursive definition for the following sets.

(a) The set of positive even integers.

Basis Step: $2 \in S$

Recursive Step: $x \in S \to x+2 \in S$.

(b) The set of negative odd integers.

Basis Step: $-1 \in S$

Recursive Step: $x \in S \to x-2 \in S$.

(c) The set of positive integers that are not divisible by 4.

Basis Step: $1, 2, 3 \in S$

Recursive Step: $x \in S \to x+4 \in S$.

(d) The set of integers that are of the form $2 \cdot 3^k$ for some nonnegative integer k.

Basis Step: $2 \in S$

Recursive Step: $x \in S \to 3x \in S$.

(e) The set of ordered pairs (a, b) such that a, b are non-negative integers and $a - b$ is odd.

Basis Step: $(1, 0), (0, 1) \in S$

Recursive Step:

- $(x, y) \in S \to (x + 2, y) \in S$
- $(x, y) \in S \to (x, y + 2) \in S$
- $(x, y) \in S \to (x + 1, y + 1) \in S$

4. Find the mistake(s) in the following proofs:

Suppose the set S is recursively defined by:

- $2 \in S$
- $x \in S \to 5x + 1 \in S$.

Prove by structural induction that each element in S is of the form $3k + 2$ for some nonnegative integer k.

(a) **Proof:**

Basis Step: The initial population 2 has the form of $3k + 2$ for some nonnegative integer k, since $2 = 3 \cdot 0 + 2$.

Recursive Step: Assume x is an arbitrary element in S. Then $x = 3k+2$ for some nonnegative integer k.

Applying the recursive definition to x, we obtain $5x + 1 = 5(3k + 2) + 1 = 15k + 11$.

We have proved that the direct descendant of x is in the form $3k + 2$ for some nonnegative integer k.

Thus, by the Principle of Structural Induction, all elements of S are in the form $3k + 2$ for some nonnegative integer k.

The author makes a conclusion in the recursive hypothesis about x being in the form $3k + 2$ for some nonnegative integer k before actually proving it. If x were in the form $3k+2$ for some nonnegative integer k, then we would not have to prove anything.

Good: Assume x is an arbitrary element in S such that $x = 3k + 2$ for some nonnegative integer k.

In addition, the author has never shown that the direct descendant $15k + 11$ of x is in the form $3k + 2$ for some nonnegative integer k.

Good: $15k+11 = 3\cdot(5k+3)+2$, where $5k+3$ is an integer. So, the descendant of x is in the required form.

(b) **Proof:**

Basis Step: The initial population 2 has the form of $3k + 2$ for some nonnegative integer k, since $2 = 3 \cdot 0 + 2$. If 2 is in S, then 11 is in S. $11 = 3 \cdot 3 + 2$, therefore 11 can be written in the form $3k + 2$

for $k = 3$ which shows that 11 has the required form.

Recursive Step: Assume x is an arbitrary element in S such that $x = 3k + 2$ for some nonnegative integer k.

Then $5x + 1 = 5(3k + 2) + 1 = 15k + 11 = 3 \cdot (5k + 3) + 2$.

Since $5k+3$ is a nonnegative integer, we have proved that the direct descendant of x is in the form $3k+2$ for some nonnegative integer k.

Thus, by the Principle of Structural Induction, all elements of S are in the form $3k+2$ for some nonnegative integer k.

Proving that the first generation of elements are in the form $3k + 2$ does not belong in the Basis Step. In fact, it does not belong anywhere in the proof. The Recursive Step takes care of the proof that the first generation of elements are in the required form, if it is properly set up. In the Basis Step, we only show that the initial elements have the given property.

(c) **Proof:**

Basis Step: The initial population 2 has the form of $3k + 2$ for some nonnegative integer k, since $2 = 3 \cdot 0 + 2$.

Recursive Step: Assume x is an arbitrary element in S such that $x = 3k + 2$ for some nonnegative integer k.

If 2 is in S, then $5 \cdot 2 + 1 = 11$ is in S and $11 = 3 \cdot 3 + 2$.

If 11 is in S, then $5 \cdot 11 + 1 = 56$ is in S and $56 = 3 \cdot 14 + 2$ and so on.

We see that all descendants are in the form $3k + 2$ for some nonnegative integer k, and hence the statement is proved by the Principle of Structural Induction.

In Recursive Step, the author only shows that the direct descendants of 2, 11 and 56 have the property, but the property has never been proved for all elements of S. The author uses examples to justify a universally quantified statement. The Principle of Structural Induction has never been carried through in the proof.

(d) **Proof:**

Basis Step: The initial population 2 has the form of $3k + 2$ for some nonnegative integer k, since $2 = 3 \cdot 0 + 2$.

Recursive Step: Assume x is an arbitrary element in S such that $x = 3k + 2$ for some nonnegative integer k. We need to show that the descendant $5x + 1$ is also in S and has the required form.

Then $5x + 1 = 5(3k + 2) + 1 = 15k + 11 = 3 \cdot (5k + 3) + 2$.

Since $5k + 3$ is a nonnegative integer, we have proved that $5x + 1$, the descendant of x, is in S and has the required form.

Thus, by Principle of Structural Induction, we have proved that all elements of S are in the form $3k + 2$ for some nonnegative integer k.

In the Recursive Step we do not show that the direct descendant of x, $5x+1$ is in S. We know that from the recursive definition of S. We only need to show that $5x+1$ is in the required form for each $x \in S$.

(e) **Proof:**

Let $P(x)$ denote the statement that $x = 3k+2$ for some integer k when x is in S.

Basis Step: $P(2)$ is true since $2 = 3\cdot 0+2$ which shows that the initial population has the required form.

Recursive Step: Assume $P(x)$ is true for an arbitrary element x in S. We need to show $P(x+1)$, that is, the descendant $5(x+1)+1$ can also be written in the form $3k+2$ for some positive integer k.

$5(x+1)+1 = 5x+6 = 3(\frac{5}{3}x + \frac{4}{3}) + 2$ which proves $P(x+1)$.

Thus, by the Principle of Structural Induction, the statement $P(x)$ is proved for all elements x in S.

There are several conceptual mistakes in this proof. The property P is not inherited from x to $5(x+1)+1$, it is inherited from x to $5x+1$ according to the recursive definition of S. In the Recursive Step, $P(x) \to P(5x+1)$ needs to be justified for all $x \in S$, not $P(x) \to P(x+1)$.
Finally, the author is trying to make the proof work by making an incorrect conclusion about $5(x+1)+1$ being in the form $3k+2$ for some integer k. $\frac{5}{3}x + \frac{4}{3}$ is not even an integer.

In general, "$P(n)$ implies $P(n+1)$" has no place in a structural induction proof, since a recursively defined set does not necessarily follow a linear structure.

Good proof:

Basis Step: The initial population 2 has the form of $3k+2$ for some nonnegative integer k, since $2 = 3\cdot 0 + 2$.

Recursive Step: Assume x is an arbitrary element in S such that $x = 3k+2$ for some nonnegative integer k.

Then $5x+1 = 5(3k+2)+1 = 15k+11 = 3\cdot(5k+3)+2$.

Since $5k+3$ is a nonnegative integer, we have proved that the direct descendant of x is in the form $3\ell+2$ for some integer ℓ.

Thus, the Principle of Structural Induction guarantees that all elements of S are in the form $3\ell+2$ for some nonnegative integer ℓ.

5. Find the mistake(s) in the following proof:

Assume S is a recursively defined set of ordered pairs defined by the following properties:

- $(1,3) \in S$
- $(a,b) \in S \to (a+1, b+3) \in S$

Use structural induction to prove that for all members (a,b) of S, $3a \leq b$.

Proof:

Basis Step: The initial population $(1,3)$ satisfies the property since $3\cdot 1 \leq 3$.

Recursive Step: Assume that (a, b) is an arbitrary member of S such that $3 \cdot a \leq b$. By the recursive hypothesis

$3(a+1) \leq b+3 = 3a \leq b$, which implies the descendant $(a+1, b+3)$ has the required property.

Thus, $3 \cdot a \leq b$ for all elements (a, b) of S by the Principle of Structural Induction.

The notation $3(a+1) \leq b+3 = 3a \leq b$ is incorrect. It does not mean that the inequalities $3(a+1) \leq b+3$ and $3a \leq b$ are equivalent. Since an equal sign is placed between $b+3$ and a, it implies that $b+3 = 3a$, which is false.

Do not put an equal sign between equivalent inequalities. To indicate equivalence between inequalities, use the double arrow sign $\leftrightarrow$, or simply just say it.

Good: Since $3a \leq b \leftrightarrow 3(a+1) \leq b+3$, the descendant $(a+1, b+3)$ has the required property.

Good: Since $3a \leq b$ is equivalent to $3(a+1) \leq b+3$, the descendant $(a+1, b+3)$ has the required property.

Good: Since $3a \leq b$, adding 3 to both sides of the inequality, we obtain $3a+3 \leq b+3$, or equivalently $3(a+1) \leq b+3$. Hence, the descendant $(a+1, b+3)$ has the required property.

Good: $3(a+1) = 3a+3 \leq b+3$, since $3a \leq b$. Hence, the descendant $(a+1, b+3)$ has the required property.

6. Find the mistake(s) in the following proof:

Assume S is a recursively defined set of ordered pairs defined by the following properties:

- $(1, -5) \in S$
- $(a, b) \in S \rightarrow (a+1, b+2) \in S$

Use structural induction to prove that for all members (a, b) of S, $3a + 2b$ is a multiple of 7.

Proof:

Basis Step: The initial population $(1, -5)$ satisfies the property since $3 \cdot 1 + 2 \cdot (-5) = -7$ and -7 is a multiple of 7.

Recursive Step: Assume that (a, b) is an arbitrary member of S such that $3a + 2b$ is a multiple of 7, that is, $3a + 2b = 7k$ for some integer k. By the recursive hypothesis

$(a+1, b+2) = 3(a+1) + 2(b+2) = (3a+2b) + 7 = 7k + 7 = 7(k+1)$, which implies that the descendant $(a+1, b+2)$ is a multiple of 7.

Thus, $3a + 2b$ is a multiple of 7 for all elements (a, b) of S by the Principle of Structural Induction.

Do not place an equal sign between the descendant and its property. For example, a puppy is not the same as the puppy's property, "having blue eyes".

The ordered pair $(a+1, b+2)$ is not a number, and $(3a+2b)+7$ is a number, so they are not equal. This is a mistake in mathematics as well as in programming.

Also the conclusion "the descendant $(a+1, b+2)$ is a multiple of 7" is incorrect. The descendant $(a+1, b+2)$ is an ordered pair, so it can not be a multiple of 7.

Good: For the descendant $(a+1, b+2)$, $3(a+1)+2(b+2) = (3a+3)+(2b+4) = (3a+2b)+7 = (7k)+7 = 7(k+1)$. Thus, we have shown that the property holds for the direct descendant of (a, b), i.e., $3(a+1)+2(b+2)$ is a multiple of 7.

7. Suppose the set S is recursively defined by:

   - $1 \in S$
   - $x \in S \to 5x - 1 \in S$
   - $x \in S \to 7x \in S$
   - $x \in S \to x^2 \in S$.

   Prove by structural induction that $x \bmod 3 = 1$ for all $x \in S$.

   Proof: We use the fact that for an integer x, $x \bmod 3 = 1$ if and only if $x = 3k+1$ for some integer k.

   Basis Step: The initial population 1 has the form $3k+1$ for some integer k because $1 = 3 \cdot 0 + 1$.

   Recursive Step: Suppose we already know about some $x \in S$ that $x = 3k+1$ for some integer k. Then

   - $5x - 1 = 5(3k+1) - 1 = 3(5k+1) + 1$
   - $7x = 7(3k+1) = 3(7k+2) + 1$
   - $x^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3k(3k+2) + 1$.

   Since k was an integer, $5k+1$, $7k+2$ and $3(3k+2)$ are integers as well; hence we have shown that the three direct descendants of x have the form $3k+1$ for some integer k.

   By the Principle of Structural Induction, we have shown that all elements of S have the form $3k+1$ for some integer k.

8. Suppose the set S is recursively defined by:

   - $2 \in S$
   - $x \in S \to 2x + 1 \in S$
   - $x \in S \to x^2 + 1 \in S$.

   Prove by structural induction that $x \bmod 3 = 2$ for all $x \in S$.

   Proof: We use the fact that for an integer x, $x \bmod 3 = 2$ if and only if $x = 3k+2$ for some integer k.

   Basis Step: The initial population 2 has the form $3k+2$ for some integer k because $2 = 3 \cdot 0 + 2$.

   Recursive Step: Suppose we already know about some $x \in S$ that $x = 3k+2$ for some integer k. Then

   - $2x + 1 = 2(3k+2) + 1 = 3(2k+1) + 2$
   - $x^2 + 1 = (3k+2)^2 + 1 = 9k^2 + 12k + 5 = 3(3k^2 + 4k + 1) + 2$.

   Since $2k+1$ and $3k^2+4k+1$ are integers, we have shown that the two direct descendants of x have the form $3k+2$ for some integer k.

   By the Principle of Structural Induction, we have shown that all elements of S have the form $3k+2$ for some integer k.

9. Suppose the set S is recursively defined by:

   - $2 \in S$
   - $x \in S \to x^2 \in S$

- $x \in S \wedge y \in S \rightarrow xy^3 \in S$.

Prove by structural induction that each member of S is a power of 2, i.e., it is equal to 2^k for some positive integer k.

Proof:

Basis Step: The initial population $2 = 2^1$ is a power of 2.

Recursive Step: Suppose that we already know about some $x, y \in S$ that $x = 2^p$ and $y = 2^q$ for some positive integers p, q. Then

- $x^2 = (2^p)^2 = 2^{2p}$ and
- $xy^3 = 2^p \cdot (2^q)^3 = 2^{p+3q}$.

Since p and q are positive integers, so are $2p$ and $p + 3q$. Thus, we have shown that all direct descendants of x and y are powers of 2.

By the Principle of Structural Induction, we have shown that all elements of S are powers of 2.

10. Suppose the set S is recursively defined by:

- $2 \in S$
- $3 \in S$
- $x \in S \rightarrow x^{-2} \in S$
- $x \in S \wedge y \in S \rightarrow xy \in S$.

Prove by structural induction that each member of S is of the form of $2^k 3^l$ for some integers k and l.

Proof:

Basis Step: The initial population $2 = 2^1 3^0$ and $3 = 2^0 3^1$ is in the required form.

Recursive Step: Suppose that we already know about some $x, y \in S$ that $x = 2^a 3^b$ and $y = 2^c 3^d$ for some integers a, b, c, d. Then

- $x^{-2} = (2^a 3^b)^{-2} = 2^{-2a} 3^{-2b}$ and
- $xy = (2^a 3^b) \cdot (2^c 3^d) = 2^{a+c} 3^{b+d}$.

Since a, b, c and d are integers, so are $-2a$, $-2b$, $a+c$ and $b+d$. Thus, we have shown that all direct descendants of x and y are product of integer powers of 2 and 3.

By the Principle of Structural Induction, we have shown that all elements of S are the product of integer powers of 2 and 3.

11. Suppose the set S is recursively defined by:

- $(0, 3) \in S$
- $(x, y) \in S \rightarrow (x + 2, y - 1) \in S$
- $(x, y) \in S \rightarrow (x - 3, y) \in S$

Prove by structural induction that $x - y \bmod 3 = 0$ for all $(x, y) \in S$.

Proof:

Basis Step: The initial population $(0, 3)$ has the property $0 - 3 \bmod 3 = 0$.

Recursive Step: Suppose that we already know that some $(x, y) \in S$ has the property $x - y \bmod 3 = 0$. By definition $x - y = 3k$ for some integer k. Then

- for the descendant $(x + 2, y - 1)$, $(x + 2) - (y - 1) = (x - y) + 3 = 3k + 3 = 3(k + 1)$ which means $(x + 2) - (y - 1) \bmod 3 = 0$ and
- for the descendant $(x-3, y)$, $(x-3)-y = (x-y)-3 = 3k-3 = 3(k-1)$ which means $(x-3)-y \bmod 3 = 0$.

By the Principle of Structural Induction, we have shown that $x - y \bmod 3 = 0$ for all $(x, y) \in S$.

12. Suppose the set S is recursively defined by:

    - $(0, 1) \in S$
    - $(x, y) \in S \rightarrow (x + 1, y - 1) \in S$
    - $(x, y) \in S \rightarrow (x + 2, y) \in S$

    Prove by structural induction that $x - y$ is odd for all $(x, y) \in S$.

    Proof:

    Basis Step: for the he initial population $(0, 1)$, $0 - 1 = -1$ which is odd.

    Recursive Step: suppose that we already know that some $(x, y) \in S$ has the property that $x - y$ is odd. By definition $x - y = 2k + 1$ for some integer k. Then

    - for the descendant $(x + 1, y - 1)$, $(x + 1) - (y - 1) = (x - y) + 2 = (2k + 1) + 2 = 2(k + 1) + 1$ which means $(x + 1) - (y - 1)$ is odd and
    - for the descendant $(x + 2, y)$, $(x + 2) - y = (x - y) + 2 = (2k + 1) + 2 = 2(k + 1) + 1$ which means $(x + 2) - y$ is odd.

    By the Principle of Structural Induction, we have shown that $x - y$ is odd for all $(x, y) \in S$.

13. Suppose B is the set of bit strings recursively defined by:

    - $010 \in B$
    - $b \in B \rightarrow 00b \in B$
    - $b \in B \rightarrow b1 \in B$

    Prove by structural induction that all members of B contain an even number of zeros.

    Proof:

    Basis Step: for the he initial population 010 has two zeros and two is an even number.

    Recursive Step: suppose that we already know that some $b \in B$ contains an even number of zeros. Let us assume that b contains $2k$ number of zeros for some integer k. Then

    - the descendant $00b$ contains $2k + 2 = 2(k + 1)$ zeros and
    - the descendant $b1$ contains $2k$ zeros.

    In both cases the descendants contain an even number of zeros.

    By the Principle of Structural Induction, we have shown all members of B contain an even number of zeros.

14. Suppose B is the set of bit strings recursively defined by:

    - $01 \in B$
    - $b \in B \to 0b \in B$
    - $b \in B \to b1 \in B$

    Prove by structural induction that B does not contain any strings in which a 0 occurs to the right of a 1.

    Proof:

    Basis Step: the initial population 01 has no 0 which occurs to the right of the 1.

    Recursive Step: suppose that we already know that some $b \in B$ does not contain a 0 which occurs to the right of a 1. Then

    - the descendant $0b$ can not contain a 0 to the right of a 1, since that 1 and 0 would be the part of the string b and b does not have that property. Similarly,
    - the descendant $b1$ can not contain a 0 to the right of a 1, since that 1 and 0 would be the part of the string b and b does not have that property.

    In both cases the descendants do not contain a 0 which occurs to the right of a 1.

    By the Principle of Structural Induction, we have shown that all members of B do not contain a 0 which occurs to the right of a 1.