Concepts:

- State the Principle of Mathematical Induction.
- Use "Basis Step" and "Inductive Step" structure in rigorous proofs.
- Write rigorous proofs by induction of summation formulas, divisibility statements, inequalities and equalities between two mathematical expressions.
- Identify common mistakes in incorrect proofs.

Notes:

- Suppose $P(n)$ represents a statement for every positive integer n. In your proof make sure $P(n)$ represents the statement and not an algebraic expression about the statement being made.
- For example: Let $P(n)$ denote the statement $\sum_{k=0}^{n-1} 2 \cdot 3^k = 3^n - 1$ for $n \geq 1$. $P(n)$ is neither $\sum_{k=0}^{n} 2 \cdot 3^k$ nor $3^n - 1$. $P(n)$ is a proposition valued function (predicate), whose output value is either true or false for each input n. In this particular case $P(n)$ is true for all positive integer n which can be proved by induction.
- Declare $P(n)$ at the beginning of the proof.
- Make sure that you always state the inductive hypothesis, and you always indicate the step in which you recall the inductive hypothesis.

Problems:

1. Complete the blanks in the following paragraph to prove the statement $P(n)$ for all positive integers n.

   - To prove a mathematical statement $P(n)$ for all positive integers n, first we verify $P(1)$. This step is called ...
   - Then we verify that the conditional statement ... for all positive integers n.
     This step is called ...
   - To prove the statement $\forall n(P(n) \to P(n+1))$, we assume ... has been proved for an arbitrary positive integer n and prove ...
   - After proving the previous step, the statement $P(n) \to P(n+1)$ is justified by for all n by the rule of inference called ...
   - Finally, we make a conclusion that the statement $P(n)$ for all $n \geq 1$ is proved by the ...

2. Find the mistake(s) in the following proofs:

   (a) Prove that $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ for all positive integers n.

   Proof: Let $P(n)$ denote the statement $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ for all positive integers n.

Basis Step: $P(1)$ is true since $\sum_{k=0}^{0} 2^k = 1 = 2^1 - 1$.

Inductive Step: Assume $P(n)$ has been proved for all positive integers n.

Then $\sum_{k=0}^{n} 2^k = \sum_{k=0}^{n-1} 2^k + 2^n = 2^n - 1 + 2^n = 2 \cdot 2^n - 1 = 2^{n+1} - 1$ which proves $P(n+1)$.
We have proved that $P(n) \to P(n+1)$ for all positive integers n.

Thus, by the Principle of Mathematical Induction $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ for all positive integers n. ■

(b) Prove that $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all positive integers n.

Proof: Let $P(n) = \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Basis Step: We verify $P(1) = \sum_{k=1}^{1} k^3 = 1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2$, which is true.

Inductive Step: Assume $P(n)$ has been proved for an arbitrary positive integer n. That is, $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Then $\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} =$
$= \frac{(n+1)^2}{4}(n^2 + 4n + 4) = \frac{(n+1)^2}{4}(n+2)^2 = \left(\frac{((n+1)(n+2)}{2}\right)^2$ which proves $P(n+1)$.

Thus, by the Principle of Mathematical Induction $P(n)$ is true for all positive integers n. ■

(c) Prove that $\sum_{k=1}^{n} (4k+3) = 2n^2 + 5n$ for all positive integers n.

Proof:

Basis Step: Since $\sum_{k=1}^{1} (4k+3) = 7 = 2 \cdot 1^2 + 5 \cdot 1$, the $n = 1$ case of the statement is true.

Inductive Step: Assume $P(n)$ has been proved for an arbitrary positive integer n.

Then $\sum_{k=1}^{n+1} (4k+3) = \sum_{k=1}^{n} (4k+3) + (n+1) = 2n^2 + 5n + (n+1) = 2n^2 + 6n + 1 = 2(n+1)^2 + 5(n+1)$ which proves $P(n+1)$.

Thus, by the Principle of Mathematical Induction, we have proved that $\sum_{k=1}^{n} (4k+3) = 2n^2 + 5n$ for all positive integers n. ■

3. Use induction to prove that $\sum_{k=1}^{n} (6k-1) = 3n^2 + 2n$ for all positive integers n.

4. Use induction to prove that $2\sum_{k=2}^{n-1} 3^k = 3^n - 9$ for all integers $n \geq 3$.

5. Use induction to prove that $\sum_{k=2}^{n-1} 4 \cdot 5^k = 5^n - 25$ for all integers $n \geq 3$.

6. Use induction to prove that $3^n < n!$ for all integers n, $n \geq 7$.

7. Use induction to prove that $\sum_{i=1}^{n} \frac{1}{i} < \frac{n}{2} + 1$ for all positive integers n.

8. Use induction to prove that 6 divides $9^n - 3^n$ for all integers $n \geq 0$.

9. Use induction to prove that 5 divides $4^{2n+1} - 3^{4n+2}$ for all integers $n \geq 0$.

10. Use induction to prove that, if S is a set with $n \geq 2$ elements, then S has $\frac{n(n-1)}{2}$ subsets containing exactly 2 elements.

Solution:

1. Complete the blanks in the following paragraph to prove the statement $P(n)$ for all positive integers n.

   - To prove a mathematical statement $P(n)$ for all positive integers n, first we verify $P(1)$. This step is called the Basis Step.
   - Then we verify that the conditional statement $P(n) \to P(n+1)$ for all positive integers n.
     This step is called the Inductive Step.
   - To prove the statement $\forall n(P(n) \to P(n+1))$, we assume $P(n)$ has been proved for an arbitrary positive integer n and prove $P(n+1)$.
   - After proving the previous step, the statement $P(n) \to P(n+1)$ is justified by for all n by the rule of inference called Univeral Generalization.
   - Finally, we make a conclusion that the statement $P(n)$ for all $n \geq 1$ is proved by the Principle of Mathematical Induction.

     The Principle of Mathematical Induction involves the justification of the Basis Step and the Inductive Step. It is possible that $P(1)$ is false and $\forall n \geq 1(P(n) \to P(n+1))$ is true. In this case the Principle of Mathematical Induction can not be applied and we can not conclude that $P(n)$ is true for all $n \geq 1$.

2. Find the mistake(s) in the following proofs:

   (a) Prove that $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ for all positive integers n.

   Proof: Let $P(n)$ denote the statement $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ for all positive integers n.

   Basis Step: $P(1)$ is true, since $\sum_{k=0}^{0} 2^k = 1 = 2^1 - 1$.

   Inductive Step: Assume $P(n)$ has been proved for all positive integers n.

   Then $\sum_{k=0}^{n} 2^k = \sum_{k=0}^{n-1} 2^k + 2^n = 2^n - 1 + 2^n = 2 \cdot 2^n - 1 = 2^{n+1} - 1$ which proves $P(n+1)$.
   We have proved that $P(n) \to P(n+1)$ for all positive integers n.

   Thus, by the principle of induction $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ for all positive integers n. ■

   The author of this proof assumes the conclusion that the statement has been proved for ALL positive integers in the inductive hypothesis. If $P(n)$ has been proved, then we don't have to prove anything.

   We assume that the statement $P(n)$ has been proved for some ARBITRARY positive integer n and prove $P(n+1)$. Then, $P(n) \to P(n+1)$ is true for all positive integers n by the rule of inference called Universal Generalization studied earlier in the course.

   (b) Prove that $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all positive integers n.

   Proof: Let $P(n) = \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Basis Step: We verify $P(1) = \sum_{k=1}^{1} k^3 = 1^3 = 1 = \left(\frac{(1(1+1)}{2}\right)^2$, which is true.

Inductive Step: Assume $P(n)$ has been proved for an arbitrary positive integer n. That is, $\sum_{k=1}^{n} k^3 = \left(\frac{(n(n+1)}{2}\right)^2$.

Then $\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \left(\frac{(n(n+1)}{2}\right)^2 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} =$
$= \frac{(n+1)^2}{4}(n^2+4n+4) = \frac{(n+1)^2}{4}(n+2)^2 = \left(\frac{((n+1)(n+2)}{2}\right)^2$, which proves $P(n+1)$.

Thus, by the Principle of Mathematical Induction $P(n)$ is true for all positive integers n. ■

The author of this proof abuses the $P(n)$ notation to identify $P(n)$ with an algebraic expression. $P(n)$ is neither $\sum_{k=1}^{n} k^3$ nor $\left(\frac{(n(n+1)}{2}\right)^2$. $P(n)$ represents the statement $\sum_{k=1}^{n} k^3 = \left(\frac{(n(n+1)}{2}\right)^2$ for each positive integer n.

The author makes the same mistake in the Basis Step. $P(1)$ represents the true statement $\sum_{k=1}^{1} k^3 = \left(\frac{(1(1+1)}{2}\right)^2$, and $P(1)$ is not 1.

(c) Prove that $\sum_{k=1}^{n} (4k+3) = 2n^2 + 5n$ for all positive integers n.

Proof:

Basis Step: Since $\sum_{k=1}^{1} (4k+3) = 7 = 2 \cdot 1^2 + 5 \cdot 1$, the $n = 1$ case of the statement is true.

Inductive Step: Assume $P(n)$ has been proved for an arbitrary positive integer n.

Then $\sum_{k=1}^{n+1} (4k+3) = \sum_{k=1}^{n} (4k+3) + (n+1) = 2n^2 + 5n + (n+1) = 2n^2 + 6n + 1 = 2(n+1)^2 + 5(n+1)$, which proves $P(n+1)$.

Thus, by the Principle of Mathematical Induction, we have proved that $\sum_{k=1}^{n} (4k+3) = 2n^2 + 5n$ for all positive integers n. ■

In the Inductive Step the author of this proof added $n+1$ to the summation instead of the $(n+1)$-term, $4(n+1)+3$. After that, the algebraic manipulation is incorrect, and the author is bluffing to obtain the correct formula for the $n+1$ case of the statement.

Also, the author refers to $P(n)$ in the inductive hypothesis and $P(n+1)$ in the Inductive Step, and $P(n)$ was never defined at the beginning of the proof.

3. Use induction to prove that $\sum_{k=1}^{n} (6k-1) = 3n^2 + 2n$ for all positive integers n.

Proof:

Basis Step:. For $n = 1$ the statement is true, since $\sum_{k=1}^{1} (6k-1) = 5$ and $3 \cdot 1^2 + 2 \cdot 1 = 5$ which implies $\sum_{k=1}^{1} (6k-1) = 1^2 + 2 \cdot 1$.

Inductive Step: Assume $\sum_{k=1}^{n}(6k-1) = 3n^2 + 2n$ for an arbitrary positive integer n.

(We wish to prove that $\sum_{k=1}^{n+1}(6k-1) = 3(n+1)^2 + 2(n+1)$.)

By the inductive hypothesis,

$\sum_{k=1}^{n+1}(6k-1) = \sum_{k=1}^{n}(6k-1) + 6(n+1) - 1 = 3n^2 + 2n + 6(n+1) - 1 = 3n^2 + 6n + 3 + 2n + 2 = 3(n+1)^2 + 2(n+1)$, which proves the $n+1$ case of the statement.

By the Principle of Mathematical Induction we have proved that $\sum_{k=1}^{n}(6k-1) = 3n^2 + 2n$ for all positive integers n. ■

4. Use induction to prove that $2\sum_{k=2}^{n-1} 3^k = 3^n - 9$ for all integers $n \geq 3$.

Proof:

Basis Step: The $n = 3$ case is true, since $2\sum_{k=2}^{2} 3^k = 18$ and $3^3 - 9 = 18$, and hence $2\sum_{k=2}^{2} 3^k = 3^3 - 9$.

Inductive Step: Assume $2\sum_{k=2}^{n-1} 3^k = 3^n - 9$ for an arbitrary integer $n \geq 3$.

(We wish to prove that $2\sum_{k=2}^{n} 3^k = 3^{n+1} - 9$.)

By the inductive hypothesis,

$2\sum_{k=2}^{n} 3^k = 2\sum_{k=2}^{n-1} 3^k + 2 \cdot 3^n = 3^n - 9 + 2 \cdot 3^n = 3 \cdot 3^n - 9 = 3^{n+1} - 9$, which proves the $n+1$ case of the statement.

By the Principle of Mathematical Induction we have proved that $2\sum_{k=2}^{n-1} 3^k = 3^n - 9$ for all integers $n \geq 3$. ■

5. Use induction to prove that $\sum_{k=2}^{n-1} 4 \cdot 5^k = 5^n - 25$ for all integers $n \geq 3$.

Proof: Let $P(n)$ represent the statement $\sum_{k=2}^{n-1} 4 \cdot 5^k = 5^n - 25$ where n is a positive integer.

Basis Step: $P(3)$ is true, since $\sum_{k=2}^{2} 4 \cdot 5^k = 100$ and $5^3 - 25 = 100$, which proves $P(3)$.

Inductive Step: Assume $P(n)$ has been proved for an arbitrary integer $n \geq 3$.

(We wish to prove $P(n+1)$, that is, $\sum_{k=2}^{n} 4 \cdot 5^k = 5^{n+1} - 25$.)

By the inductive hypothesis,

$\sum_{k=2}^{n} 4 \cdot 5^k = \sum_{k=2}^{n-1} 4 \cdot 5^k + 4 \cdot 5^n = 5^n - 25 + 4 \cdot 5^n = 5^{n+1} - 25$, which proves $P(n+1)$.

By the Principle of Mathematical Induction we conclude that $\sum_{k=2}^{n-1} 4 \cdot 5^k = 5^n - 25$ for all integers $n \geq 3$. ■

6. Use induction to prove that $3^n < n!$ for all integers n, $n \geq 7$.

Proof:

Basis Step: The $n = 7$ case, $3^7 < 7!$ is true, since $2187 < 5040$.

Inductive Step: Assume $3^n < n!$ has been proved for an arbitrary integer $n \geq 7$.

(We wish to prove that $3^{n+1} < (n+1)!$ is true.)

Using the inductive hypothesis and the inequality $3 < n+1$ for $n \geq 7$, we obtain
$3^{n+1} = 3^n \cdot 3 < n! \cdot 3 < n!(n+1) = (n+1)!$ which proves that $P(n+1)$ is true.

By the Principle of Mathematical Induction we have proved that $3^n < n!$ for all integers $n \geq 7$. ■

7. Use induction to prove that $\sum_{i=1}^{n} \frac{1}{i} < \frac{n}{2} + 1$ for all positive integers n.

   Proof: Let $P(n)$ denote the statement $\sum_{i=1}^{n} \frac{1}{i} < \frac{n}{2} + 1$ where n is a positive integer.

   Basis Step: For $n = 1$, $P(1)$ is true, since $\sum_{i=1}^{1} \frac{1}{i} = 1 < \frac{2}{2} + 1 = 1.5$.

   Inductive Step: Assume $P(n)$ has been proved for an arbitrary positive integer n.

   (We wish to prove $P(n+1)$, that is, $\sum_{i=1}^{n+1} \frac{1}{i} < \frac{n+1}{2} + 1$ is true.)

   Using the inductive hypothesis and the inequality $\frac{1}{n+1} \leq \frac{1}{2}$ for $n \geq 1$, we obtain
   $\sum_{i=1}^{n+1} \frac{1}{i} = \sum_{i=1}^{n} \frac{1}{i} + \frac{1}{n+1} < \frac{n}{2} + 1 + \frac{1}{n+1} < \frac{n}{2} + 1 + \frac{1}{2} = \frac{n+1}{2} + 1$, which proves $P(n+1)$.

   By the Principle of Mathematical Induction we have proved the statement $\sum_{i=1}^{n} \frac{1}{i} < \frac{n}{2} + 1$ for all positive integers n. ■

8. Use induction to prove that 6 divides $9^n - 3^n$ for all integers $n \geq 0$.

   Proof:

   Basis Step: The $n = 0$ case is true since 6 divides $9^0 - 3^0 = 0$ by the definition of divisibility.

   Inductive Step: Assume 6 divides $9^n - 3^n$ for an arbitrary positive integer n. By the definition of divisibility, $9^n - 3^n = 6k$ for some integer k.

   (We wish to prove that 6 divides $9^{n+1} - 3^{n+1}$.)

   Using the inductive hypothesis and some algebraic manipulation,
   $9^{n+1} - 3^{n+1} = 9 \cdot 9^n - 3 \cdot 3^n = (6+3) \cdot 9^n - 3 \cdot 3^n = 6 \cdot 9^n + 3 \cdot (9^n - 3^n) = 6 \cdot 9^n + 3 \cdot (6k) = 6 \cdot (9^n + 3k)$.
   Since $9^n + 3k$ is an integer, $9^{n+1} - 3^{n+1}$ is divisible by 6.

   Thus, the proof is completed by induction. ■

9. Use induction to prove that 5 divides $4^{2n+1} - 3^{4n+2}$ for all integers $n \geq 0$.

Proof: Let $P(n)$ denote the statement that 5 divides $4^{2n+1} - 3^{4n+2}$ where n is an integer $n \geq 0$.

Basis Step: $P(0)$ is true, since 5 divides $4^1 - 3^2 = -5$ by the definition of divisibility.

Inductive Step: Assume $P(n)$ has been proved for an arbitrary positive integer n. By the definition of divisibility, $4^{2n+1} - 3^{4n+2} = 5k$ for some integer k.

(We wish to prove $P(n+1)$, that is 5 divides $4^{2(n+1)+1} - 3^{4(n+1)+2}$.)

Using the inductive hypothesis and some algebraic manipulation, $4^{2(n+1)+1} - 3^{4(n+1)+2} = 16 \cdot 4^{2n+1} - 81 \cdot 3^{4n+2} = (81-65) \cdot 4^{2n+1} - 81 \cdot 3^{4n+2} = 81 \cdot (4^{2n+1} - 3^{4n+2}) - 65 \cdot 4^{2n+1} = 81 \cdot (5k) - 65 \cdot 4^{2n+1} = 5 \cdot (81k - 13 \cdot 4^{2n+1})$, which proves $P(n+1)$, and hence the proof is complete by the Principle of Mathematical Induction. ■

10. Use induction to prove that, if S is a set with $n \geq 2$ elements, then S has $\frac{n(n-1)}{2}$ subsets containing exactly 2 elements.

Proof: Let $P(n)$ represent the statement "a set S with $n \geq 2$ elements has $\frac{n(n-1)}{2}$ subsets containing exactly 2 elements."

Basis Step: $P(2)$ is true, since a set with 2 elements has exactly $\frac{2(2-1)}{2} = 1$ subset with exactly 2 elements, which is itself the set.

Inductive Step: Assume $P(n)$ has been proved for an arbitrary positive integer $n \geq 2$.

(We wish to prove $P(n+1)$, that is, "a set S with $n+1$ elements has $\frac{(n+1)n}{2}$ subsets with exactly 2 elements.")

Let x be an element of S. Then $S = (S \setminus \{x\}) \cup \{x\}$. Since $S \setminus \{x\}$ has n elements, according to the inductive hypothesis, it contains $\frac{n(n-1)}{2}$ subsets with exactly 2 elements. These subsets have two elements and do not contain the element x. The subsets of size 2 that contain x are in the form of $\{x, a\}$ for some $a \in S \setminus \{x\}$. There are n such subsets of size 2. Thus, altogether the number of 2-element subsets is $n + \frac{n(n-1)}{2} = \frac{(n+1)n}{2}$, which proves $P(n+1)$.

By the Principle of Mathematical Induction we have shown that if S is a set with $n \geq 2$ elements, then S has $\frac{n(n-1)}{2}$ subsets containing exactly 2 elements. ■