Concepts:

- Distinguish between propositions and non-propositions.
- Write the truth table of logical operators negation, conjunction, disjunction, exclusive disjunction, conditional and biconditional, and compound propositions.
- Negate mathematical inequalities.
- Translate variously phrased conditional statements from English into mathematical statements and vice versa, including statements using "necessary", "sufficient", "only if" and, "unless."
- Recognize differently phrased English language conditional statements as logically equivalent.
- Recognize variations of conditional statements (inverses, converses, contrapositives) as equivalent or non-equivalent.
- Understand the order of precedence of logical operators and be able to add or delete parentheses without changing the meaning or making the Boolean expression invalid.
- Logical operators on bit strings.

Problems:

1. Check whether the following statements are true or false.
   (a) The following two conditional statements are equivalent.

   "If it is morning, then the sun rises" and *"the sun rises if it is morning"*

   (b) The mathematical equation " $a^2 + b^2 = c^2$" is a proposition for any real numbers a, b, and c.

   (c) *"a is necessary for b"* is logically equivalent to *"a only if b."*

   (d) *"a is necessary for b"* is logically equivalent to *"b is sufficient for a."*

   (e) The truth value of *"If it is raining, then the sun does not shine"* is false if *"It is not raining today."*

2. Let a and b be the bit strings: $a = 0101$ and $b = 1100$.

   Find the values of the logical bit string operations below.

   (a) a AND b

   (b) a OR b

   (c) a XOR b

   (d) NOT a

   (e) NOT b

3. Find the inverse, converse, contrapositive and the fully simplified negation of the following conditional statements. We assume x and y are real numbers.

   (a) If $x = 0$ or $y = 0$, then $x \cdot y = 0$.

   (b) If $x^2 - 3x + 2 > 0$, then $x > 1$ and $x < 2$.

4. Which of the following is/are equivalent to the expression $p \wedge q \vee r \rightarrow s$?

   (a) $(p \wedge q) \vee (r \rightarrow s)$

   (b) $(p \wedge (q \vee r)) \rightarrow s$

   (c) $p \wedge ((q \vee r) \rightarrow s)$

   (d) $p \wedge (q \vee (r \rightarrow s))$

   (e) $((p \wedge q) \vee r) \rightarrow s$

5. Let $p =$ *"I walk to school every day"*, $q =$ *"I use the swimming pool"* and $r =$ *"I build cardiovascular health."* Express the following statement as a combination of p, q and r along with necessary logic symbols.

   "To build cardiovascular health, it is sufficient for me to walk to school every day or to use the swimming pool."

   (a) $(p \vee q) \rightarrow r$

   (b) $r \rightarrow (p \vee q)$

   (c) $(r \rightarrow p) \vee q$

   (d) $p \vee (q \rightarrow r)$

   (e) none of these

6. Rewrite the following as an "if-then" statement.

   (a) For any integer n, it is necessary that n is even for n^2 to be even.
   (b) To live in the dormitory at ASU it is sufficient to be a freshman.
   (c) You see lightning only if you hear thunder.
   (d) I will get my work done on time unless I procrastinate.

7. Use truth tables to verify the absorption laws.

   (a) $p \vee (p \wedge q) \equiv p$

   (b) $p \wedge (p \vee q) \equiv p$

Solutions:

1. Check whether the following statements are true or false.

   (a) The following two conditional statements are equivalent.

   "If it is morning, then the sun rises" and *"the sun rises if it is morning"*

   True. The two statements are equivalent. The conditional statement *"if P, then Q"* can also be expressed as *"Q, if P"* or *"P only if Q."*

   (b) The mathematical equation "$a^2 + b^2 = c^2$" is a proposition.

   False. The above equation contains three undefined variables a, b, and c. The equation "$a^2 + b^2 = c^2$" has no truth value unless we specify the values of these variables.

   (c) "*a is necessary for b*" is logically equivalent to "*a only if b*."

   False. "*a is necessary for b*" is equivalent to "*if b, then a*", and "*a only if b*" is equivalent to "*if a, then b*." Thus, they are converse statements of each other, so they are not equivalent.

   (d) "a is necessary for b" is logically equivalent to "b is sufficient for a."

   True. They are both equivalent to the expression "*if b, then a*."

   (e) The truth value of "*If it is raining, then the sun does not shine*" is false if "*it is not raining today.*"

   False. A conditional statement of the form "*if P then Q*" is true when the premise P is false regardless of the truth value of the conclusion Q.

2. Let a and b be the bit strings: $a = 0101$ and $b = 1100$.

   Find the values of the logical bit string operations below.

   (a) a AND b = 0100

   (b) a OR b = 1101

   (c) a XOR b = 1001

   (d) NOT a = 1010

   (e) NOT b = 0011

   NOTE. 1 represents "True" and 0 represents "False." The logical operators are performed bit-wise between two bit strings of equal length.

3. Find the inverse, converse, contrapositive and the fully simplified negation of the following conditional statements. We assume that x and y are real numbers.

(a) If $x = 0$ or $y = 0$, then $x \cdot y = 0$.

Inverse: If $x \neq 0$ and $y \neq 0$, then $x \cdot y \neq 0$.

Converse: If $x \cdot y = 0$, then $x = 0$ or $y = 0$.

Contrapositive: If $x \cdot y \neq 0$, then $x \neq 0$ and $y \neq 0$.

Negation: $x = 0$ or $y = 0$ but (and) $x \cdot y \neq 0$.

(b) If $x^2 - 3x + 2 > 0$, then $x > 1$ and $x < 2$.

Inverse: If $x^2 - 3x + 2 \leq 0$, then $x \leq 1$ or $x \geq 2$.

Converse: If $x > 1$ and $x < 2$, then $x^2 - 3x + 2 > 0$.

Contrapositive: If $x \leq 1$ or $x \geq 2$, then $x^2 - 3x + 2 \leq 0$.

Negation: $x^2 - 3x + 2 > 0$ but (and) $x \leq 1$ or $x \geq 2$.

4. Which of the following is/are equivalent to the expression $p \wedge q \vee r \rightarrow s$?

(a) $(p \wedge q) \vee (r \rightarrow s)$

(b) $(p \wedge (q \vee r)) \rightarrow s$

(c) $p \wedge ((q \vee r) \rightarrow s)$

(d) $p \wedge (q \vee (r \rightarrow s))$

(e) $((p \wedge q) \vee r) \rightarrow s$ Correct.

NOTE. We use the order of precedence for Boolean operators.
The following precedence is conventionally used, in order from highest to lowest:
1. negation 2. conjunction 3. disjunction 4. conditional 5. biconditional

5. Let $p =$ "I walk to school every day", $q =$ "I use the swimming pool" and $r =$ "I build cardiovascular health." Express the following statement as a combination of p, q and r along with necessary logic symbols.

"To build cardiovascular health, it is sufficient for me to walk to school every day or to use the swimming pool."

(a) $(p \vee q) \rightarrow r$ Correct.

(b) $r \rightarrow (p \vee q)$

(c) $(r \rightarrow p) \vee q$

(d) $p \vee (q \rightarrow r)$

(e) none of these

NOTE. The original statement is equivalent to "if I walk to school every day or I use the swimming pool, then I will build cardiovascular health."

6. Rewrite the following as an "if-then" statement.

   (a) For any integer n, it is necessary that n is even for n^2 to be even.

   For any integer n, if n^2 is even, then n is even.

   (b) To live in the dormitory at ASU, it is sufficient to be a freshman.

   If you are a freshman, then you can live in the dormitory at ASU.

   (c) You see lightning only if you hear thunder.

   If you see lightning, then you will hear thunder.

   (d) I will get my work done on time unless I procrastinate.

   If I do not procrastinate, then I will get my work done.

7. Use truth tables to verify the absorption laws.

   (a) $p \vee (p \wedge q) \equiv p$

   (b) $p \wedge (p \vee q) \equiv p$

   We construct the following truth table:

p	q	$p \vee q$	$p \wedge q$	$p \vee (p \wedge q)$	$p \wedge (p \vee q)$
0	1	1	0	0	0
0	0	0	0	0	0
1	1	1	1	1	1
1	0	1	0	1	1

   For part (a), we can easily observe that the first and fifth columns of the above truth table are identical. Hence, part (a) is true.
   Similarly for part (b), we observe that the first and sixth columns of the above truth table are identical. Hence, part (b) is true.