MAT 267: Calculus III For Engineers

Test 2 Review

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1. Give a geometric description of the domain for each function and sketch it.

   a) $f(x, y) = \ln(1 - x^2 - y^2)$
   b) $f(x, y) = \sqrt{x} + \sqrt{y}$
   c) $f(x, y) = \cos(x^2 + y^2)$
   d) $fx, y) = \sqrt{y - x^2}/(1 - x^2)$

2. Give a geometric description of the contour curves for each function.

   a) $f(x, y) = \sqrt{4 - x - y}$
   b) $f(x, y) = 3y - x^2$
   c) $f(x, y) = x^2 - y^2$
   d) $fx, y) = (x^2 + y^2)^{3/2}$

3. Give a geometric description of the level surfaces for each 3-variable function.

   a) $F(x, y, z) = z - x^2 - y^2$
   b) $F(x, y, z) = 4(x - 2)^2 + 9(y - 1)^2 + z^2$
   c) $F(x, y, z) = (x^2 + y^2 + z^2)^{-2}$
   d) $F(x, y, z) = x + 2y - z$

4. Sketch the contour of the function $f(x, y) = 2x^2 + y^2$ corresponding to $z = 8$.

5. Find all second partial derivatives of the given functions.

   a) $f(x, y) = e^{xy} \sin(2y)$
   b) $g(x, y) = xy/(x - y)$

6. Find the equation of the tangent plane for each function at the given point.

   a) $f(x, y) = xe^{xy}$ at $(2, 0)$
   b) $g(x, y) = 2x^2 + \frac{1}{y}$ at $(3, 1)$
   c) $h(x, y) = \sqrt{y + \cos^2 x}$ at $(0, 0)$

7. Use linearization to approximate $f(2.04, 4.99)$, given that

$$f(2, 5) = 6, \quad f_x(2, 5) = -1, \quad f_y(2, 5) = 3.$$

8. Use differentials to approximate the change in the volume of a cone with radius $r = 3$ and height $h = 4$ if the radius is increased by 0.1 and the height is decreased by 0.2. The volume of a cone is $V(r, h) = \frac{\pi}{3} r^2 h$.

9. If $u = x^2 y^3 + z^4$ and $x = 3t^2 + t + 1,\ y = te^t - 2,\ z = \sin t$, find du/dt when $t = 0$.

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10. Consider a rectangle with dimensions x and y in cm. If x is growing at a rate of 2 cm/s and y is growing at a rate of 3 cm/s, find the rate at which the length of the diagonal is changing when $x = 5$ cm and $y = 8$ cm.

11. Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ at $(s, t) = (0, 2)$, given that

$$w = \sqrt{x^2 + y^2 + x^2}, \quad x = se^t, \quad y = te^s, \quad z = e^{st}.$$

12. Use the chain rule to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the implicit surface: $yz + x\ln(y) = z^2$

13. Find the rate of change of the function $f(x, y) = x^4 - x^2y^3$ at the point $(-1, 3)$ in the direction of $\theta = 2\pi/3$.

14. Find the rate of change of the function $f(x, y, z) = 10e^{-2x^2-3y^2+4z^2}$ at the point $(1, 1, 1)$ in the direction that points toward the origin.

15. Find the maximum slope of the surface $z = e^{x^2-y^2}$ at the point $(3, 3)$ and the unit vector that points in the direction of maximum increase.

16. Find all points where the minimum rate of increase of $f(x, y) = x^2 + y^2 - 2x - 4y$ is in the direction of $\mathbf{i} + \mathbf{j}$.

17. Find a normal vector to the implicit surface $x + y + z = 3e^{xyz}$ at the point $(0, 1, 2)$.

18. Find the equations of the tangent plane and normal line for the implicit surface $xyz^3 - 2z = 6$ at the point $(4, 2, 1)$.

19. Find and classify all critical points of each function using the 2nd derivative test.

    a) $f(x, y) = xy - 2x - 2y - x^2 - y^2$
    b) $f(x, y) = x^3 + 3xy + y^3$

20. Find the point(s) on the surface $z^2 = 9 + xy$ that are closest to the origin.

21. Find three numbers x, y and z that sum to 27 and whose product is at a maximum.

22. Find the dimensions of a rectangular box (with no top) that minimize the surface area if the box must have a volume of 32 ft^3.

23. Find the absolute extrema of $f(x, y) = 2xy$ over the disk $x^2 + y^2 \leq 9$.

24. Find the volume of the region that lies between the surfaces $z = 2x^2 + 3y^2 + 8$ and $z = 3$ and over the rectangle $R = [0, 1] \times [0, 3]$.

25. Evaluate the double integral $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right)\, dy\, dx$.

26. Find the average value of $f(x, y) = e^{2x}\cos(y)$ over the rectangle $R = [0, 2] \times [0, \frac{\pi}{4}]$.

27. Integrate $f(x, y) = x^2\cos(y)$ over the region bounded by $y = 0,\ y = x^3$ and $x = 1$.

28. Evaluate the double integral by switching the order of integration: $\int_0^1 \int_{2y}^2 e^{x^2}\, dx\, dy$.

29. Use a double integral to find the volume of the tetrahedron that is under the plane $x+y+z=2$ and in the first octant.

30. Evaluate $\iint_D \sin(x^2+y^2)\, dA$ where D is the region in the first quadrant between the circles $x^2+y^2=1$ and $x^2+y^2=9$.

31. Evaluate $\iint_D \tan^{-1}\left(\frac{y}{x}\right)\, dA$ where D is the region between the circles $x^2+y^2=1$ and $x^2+y^2=4$ and between the lines $y=0$ and $y=x$.

32. Use a double integral to find the area of one loop of the rose $r=\cos(3\theta)$.

33. Find the volume of the region bounded by the surfaces $z=x^2+y^2$ and $z=3-3x^2-3y^2$.

34. Evaluate the double integral by transforming to polar coordinates:

$$\int_{-2}^{0}\int_{0}^{\sqrt{4-x^2}} x\, dy\, dx$$

Answers

1. a) $D = \{(x, y) \mid x^2 + y^2 < 1\ \}$ (the region inside the unit circle, not including the circle)
   b) $D = \{(x, y) \mid x \geq 0,\ y \geq 0\ \}$ (the first quadrant, including the axes)
   c) $D = \mathbb{R}^2$ (the entire xy plane)
   d) $D = \{(x, y) \mid y \geq x^2,\ x \neq \pm 1\ \}$ (the region on or above the parabola $y = x^2$, with the lines $x = \pm 1$ removed)

2. a) parallel lines, all with slope -1
   b) parabolas with a symmetry axis along y-axis
   c) hyperbolas centered around the origin
   d) concentric circles centered around the origin, unevenly spaced

3. a) paraboloids with a symmetry axis along z-axis
   b) concentric ellipsoids centered around the point $(2, 1, 0)$
   c) concentric spheres centered around the origin, unevenly spaced
   d) parallel planes, each with normal vector $\langle 1, 2, -1\rangle$

4. An ellipse centered at the origin and passing through the points $(\pm 2, 0)$ and $(0, \pm\sqrt{8})$

5. a)
   $f_{xx} = y^2 e^{xy} \sin(2y)$
   $f_{xy} = f_{yx} = e^{xy}[(1 + x) \sin(2y) + 2y \cos(2y)]$
   $f_{yy} = e^{xy}[(x^2 - 4) \sin(2y) + 4x \cos(2y)]$

   b)
   $g_{xx} = 2y^2/(x - y)^3$
   $g_{xy} = g_{yx} = 2y(2y - x)/(x - y)^3$
   $g_{yy} = 2x^2/(x - y)^3$

6. a) $z = x + 4y$
   b) $z = 12x - y - 16$
   c) $z = 1 + \frac{1}{2}y$

7. $f(2.04, 4.99) \approx 5.93$

8. $\Delta V \approx \pi/5$

9. $\left.\frac{du}{dt}\right|_{t=0} = -4$

10. $\frac{34}{\sqrt{89}}$ cm/s

11. $\left.\frac{\partial w}{\partial s}\right|_{(0,2)} = 6/\sqrt{5},\ \left.\frac{\partial w}{\partial t}\right|_{(0,2)} = 2/\sqrt{5}$

12. $\frac{\partial z}{\partial x} = \ln(y)/(2z - y),\ \frac{\partial z}{\partial y} = 1/[y(2z - y)]$

13. $-25 - 27\sqrt{3}/2$

14. $20/(\sqrt{3}e)$

15. max slope: $6\sqrt{2}$, unit vector: $\frac{1}{\sqrt{2}}\langle 1, -1\rangle$

16. $(1/2, 3/2)$

17. $\mathbf{n} = \langle -9, 1, 1 \rangle$

18. Tangent plane: $x + 2y + 11z = 19$, Normal line: $\langle 4 + t, 2 + 2t, 1 + 11t \rangle$

19. a) $(-2, -2)$ local max
    b) $(0, 0)$ saddle, $(-1, -1)$ local max

20. $(0, 0, \pm 3)$

21. $(x, y, z) = (9, 9, 9)$

22. $(x, y, z) = (4, 4, 2)$ ft

23. abs min $= -9$ at $(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$, $(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$
    abs min $= 9$ at $(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$, $(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$

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25. $\frac{21}{2} \ln(2)$

26. $\frac{e^4 - 1}{\pi\sqrt{2}}$

27. $\frac{1}{3}(1 - \cos 1)$

28. $\int_0^2 \int_0^{x/2} e^{x^2}\ dy\ dx = \frac{1}{4}(e^4 - 1)$

29. $4/3$

30. $\frac{\pi}{4}(\cos 1 - \cos 9)$

31. $3\pi^2/16$

32. $\pi/12$

33. $9\pi/8$

34. $-8/3$