LIST OF CONCEPTS AND SKILLS FOR TEST 2
The test covers sections 3.1-3.5, 3.7, 3.8, 6.1, 6.2

Chapter 3
The Wronskian: (Section 3.2)

- Know Theorem 3.2.1: Existence and Uniqueness. Consider the Initial Value Problem
$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'$$
where $p(t)$, $q(t)$ and $g(t)$ are continuous on an open interval I that contains t_0. Then there is exactly one solution of this problem, and the solution exists throughout the interval I.
- Know Theorem 3.2.2: **Principle of Superposition**. If y_1 and y_2 are solutions of the homogeneous differential equation $y'' + p(t)y' + q(t)y = 0$, so is $c_1y_1 + c_2y_2$ for any constants c_1 and c_2.
- Know how to find the Wronskian of two functions, y_1 and y_2.
- Know Theorem 3.2.4: **Wronskian of Solutions**: Given two solutions y_1 and y_2 of a homogeneous ODE, always check whether the Wronskian of the two solutions is not everywhere zero. If this is case the two solutions are linearly independent and we say they form a Fundamental Set of Solutions for the ODE. The general solution is given by $c_1y_1 + c_2y_2$ with c_1 and c_2 arbitrary constants.

Linear Homogeneous DEs with constant coefficients (Sections 3.1, 3.3, 3.4)
Given the homogeneous ODE $ay'' + by' + cy = 0$ with a, b and c constant:

- The characteristic equation is $ar^2 + br + c = 0$.
- If the roots of the characteristic equation are real and distinct, r_1 and r_2, then the general solution of the homogeneous ODE is $y = c_1e^{r_1t} + c_2e^{r_2t}$.
- If $r_1 = r_2 = r$, then the general solution is $y = c_1e^{rt} + c_2te^{rt}$.
- If the roots of the characteristic equation, r_1 and r_2, are complex conjugates $a \pm bi$, then the general solution is $y = c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt)$.
- Know how to determine the exact solution (solve for c_k) of the above when given appropriate initial conditions.

This can be generalized to higher order differential equations:
If a root r of the characteristic equation has multiplicity k, then $e^{rt}, te^{rt}, t^2e^{rt}, \ldots, t^{k-1}e^{rt}$ are k linearly independent solutions.
If the complex roots $a \pm bi$ are repeated k times, then the general solution must contain the terms $e^{at}\cos(bt)$, $e^{at}\sin(bt)$, $te^{at}\cos(bt)$, $te^{at}\sin(bt), \ldots, t^{k-1}e^{at}\cos(bt)$, $t^{k-1}e^{at}\sin(bt)$.

Reduction of order (Section 3.4)
Suppose we know one solution $y_1(t)$ of $y'' + p(t)y' + q(t)y = 0$, not everywhere zero. To find another solution, let $y_2(t) = v(t)y_1(t)$. To find the unknown function $v(t)$, substitute $y_2(t)$ into the differential equation. The result will be a DE involving v' and v''. This DE can be reduced to first order by letting $u = v'$ and solved using techniques from chapters 1 and 2.

Non-Homogeneous Equations, Undetermined Coefficients (Section 3.5)

- Know Theorem 3.5.2: **Solutions of Nonhomogeneous Equations**: The general solution of a NonHomogeneous ODE is given by $y = y_c + y_p$ where y_c is the complementary solution (the general solution of the associated HODE) and y_p is a particular solution of the Non-Homogeneous ODE.

- Method of **undetermined coefficients** for linear DEs with constant coefficients: This method works only when the function $g(t)$ is a polynomial, an exponential function, a sine or cosine and or a sum/product of these functions. The method consists of taking as a trial solution for y_p a linear combination of linearly independent terms appearing in $g(t)$ and in all their derivatives $g'(t)$, $g''(t)$, … . If any of these terms duplicates a solution of the associated homogeneous ODE, then we need to multiply the term by t^s where s is the smallest non-negative integer such that no term in y_p duplicates a term in the complementary function y_c. The undetermined coefficients A, B, C, … are then determined by substituting y_p and the appropriate derivatives of it into the original DE. Once y_p is determined, a general solution is given by $y = y_c + y_p$ where y_c is the complementary function.
- You must be able to write the correct expression for y_p with the minimal number of undetermined coefficients.
- You must be able to solve for the undetermined coefficients and thus write explicitly a solution of the given non-homogeneous ODE.
- You must be able to find a general solution for the given ODE.

Mechanical vibrations (Section 3.7)

- Assume a mass m is attached to a spring with constant k, and assume there is a dashpot producing damping proportional to the velocity, with constant γ. When no external force is applied, the ODE governing the motion of the mass is given by $mu'' + \gamma u' + ku = 0$.
- *Undamped Free Vibration*: if $\gamma = 0$, the equation reduces to $mu'' + ku = 0$ with solution $u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$, (Simple Harmonic Motion) where $\omega_0 = \sqrt{k/m}$ is called the circular frequency. The period of the motion is given by $T = 2\pi/\omega_0$. Be able to rewrite the solution as $u(t) = R\cos(\omega_0 t - \delta)$ where R is the amplitude and δ is the phase angle.
- *Damped Free Vibration*: if $\gamma \neq 0$, we can have three different kinds of solutions depending on the roots of the characteristic equation. If the roots are both real, they must be negative and the motion is **overdamped**; if the roots are repeated (necessarily real and negative) the motion is **critically damped**; if the roots are complex (with negative real part) the motion is **underdamped**. For underdamped motion, know how to determine the quasi frequency and quasi period.

Forced Vibrations (Section 3.8)

We consider the case of mass m is attached to a spring with constant k, a dashpot with constant γ and also acted upon by an external force $F(t)$. The governing ODE is given by: $mu'' + \gamma u' + ku = F(t)$. We consider the particular case where $F(t) = F_0 \cos(\omega t)$.

- *Forced Vibrations Without Damping*: if $\gamma = 0$, the equation reduces to $mu'' + ku = F_0 \cos(\omega t)$.
  - If $\omega \neq \omega_0$ the solution is $u = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + C\cos(\omega t)$ where $u_c = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$ is the complementary solution and $u_p = C\cos(\omega t)$ is the particular solution (determined using the method of undetermined coefficients). Note that the constants c_1 and c_2 depend on the initial conditions. So we see that the resulting motion is the superposition of two oscillations, one with natural frequency ω_0, the other with frequency ω of the external force.
  - **BEATS**: If $\omega \approx \omega_0$ and the initial conditions are set to $u(0) = 0, u'(0) = 0$, we have the

phenomenon of beats (a rapid oscillation with a (comparatively) slowly varying periodic amplitude).

- **RESONANCE**: If $\omega = \omega_0$ we have the phenomenon of pure resonance, the increase without bound in the amplitude of the oscillations

- *Forced Vibrations with Damping*: if $\gamma \neq 0$, we can have three different kind of solutions depending on the roots of the characteristic equation. In any case, the complementary solutions $u_c(t) \to 0$ as $t \to \infty$. So, $u_c(t)$ is a *transient* solution (i.e. dying out in time) leaving only the particular solution $u_p = C\cos(\omega t - \alpha)$ (*steady state solution*).

NOTE: It is not a good idea to memorize the formulas above. Just remember the assumptions for Beats and Pure Resonance and be able to solve the Differential Equations using the method of Undetermined Coefficients from Section 3.5.

CHAPTER 6: THE LAPLACE TRANSFORM

6.1: Definition of the Laplace Transform

- The Laplace transform of $f(t)$ is defined through an improper integral $\int_0^\infty f(t)e^{-st}dt$. Be able to calculate the transform of basic functions using the definition.
- Know how to compute the Laplace transform of functions <u>using the table</u>. A preliminary algebraic manipulation may be necessary.

6.2: Solution of Initial Value Problems

- Know how to compute inverse transform functions. You will have to use some algebraic manipulation and/or partial fractions decomposition (PFD) to put it in a form that can be found in the Table.
- Know how to transform derivatives of functions:

(1) $\mathcal{L}\{y'(t)\} = sY(s) - y(0)$

(2) $\mathcal{L}\{y''(t)\} = s^2Y(s) - s\,y(0) - y'(0)$

- Know how to partially work to solve linear differential equations using the Laplace transform.
  - Apply the Laplace transform to both sides of the equation and use formulas (1) and (2) above so that the equation contains the Laplace of y only (which we denote by $Y(s)$).
  - Substitute the initial conditions and solve for $Y(s)$.

- Compute the inverse transform of $Y(s)$. This is the solution to the DE.

- Know how to solve homogeneous linear differential equations using the Laplace transform by inverting the transform.