MAT 275 TEST 2 **PRACTICE PROBLEMS**

I. The Wronskian.

1. Determine the longest interval in which the given initial value problem is certain to have a unique twice-differentiable solution.
   a) $t(t-4)y'' + 3ty' + 4y = 2, \quad y(6) = 0, \quad y'(6) = -1$
   b) $(t+1)y'' + ty' + y = \sec t, \quad y(0) = 2, y'(0) = -1$
   c) $(t-4)y'' + 3ty' + \ln(t)\, y = \sin t, \quad y(1) = -2, y'(1) = -1$

2. Find the Wronskian of the following pair of functions, $\{3e^{2t},\ te^{2t}\}$.

3. Consider the ODE $t^2y'' + 3ty' + y = 0$ with the initial conditions $y(1) = 1,\ y'(1) = 1$.
   (i) What is the maximum interval of validity, I, of the solution?
   (ii) Verify that the functions $y_1(t) = t^{-1}$ and $y_2(t) = t^{-1}\ln(t)$ satisfy the ODE for t in the interval I.
   (iii) Find the Wronskian $W(y_1, y_2)$ to show that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions.
   (iv) Solve the initial value problem.

4. Suppose $y_1(t) = t$ and $y_2(t) = t^2$ are both solutions of the second order linear equation $y'' + p(t)y' + q(t) = 0$. Which of the functions below are guaranteed to also be solutions of the same equation?
   A. $y = t^2 - 1$ B. $y = 5t$ C. $y = -9t^2 + 17t$ D. $y = 0$

5. Which of the following is NOT a fundamental set of solutions for $y'' - y = 0$?
   A. $\{e^t,\ e^{-t}\}$ B. $\{2e^t,\ 2e^{-t}\}$ C. $\{te^t,\ e^{-t}\}$ D. $\left\{(e^t + e^{-t}),\ \frac{1}{2}(e^t + e^{-t})\right\}$
   E. $\left\{\frac{1}{2}(e^t + e^{-t}),\ \frac{1}{2}(e^t - e^{-t})\right\}$ F. $\left\{\frac{1}{2}(e^t + e^{-t}),\ e^t\right\}$

II. HODEs/IVP with constant coefficients.

1. Find a real valued solution to the following initial value problems.
   a. $y'' - 6y' + 13y = 0$, with $y(0) = 1,\ y'(0) = 1$.
   b. $y'' + 4y' + 4y = 0$, with $y(0) = 1,\ y'(0) = -4$.
   c. $6y'' + 7y' + 2y = 0$, with $y(0) = 7,\ y'(0) = -4$.

III. Reduction of order:

1. The ODE $t^2y'' + 3ty' + y = 0$ has a solution $y_1(t) = \frac{1}{t}$ for $t > 0$. Find the general solution.
2. The ODE $2ty'' - 5y' + \frac{3}{t}y = 0$ has a solution $y_1(t) = t^3$ for $t > 0$. Find the general solution.

IV. Undetermined coefficients

1. Find a particular solution and the general solution of the ODE $y'' + 2y' + y = 3t^2 + 5e^{2t}$.
2. Find a particular solution and the general solution of the ODE: $y'' - y' - 2y = 4\sin(3t)$.
3. Find a particular solution and the general solution of the ODE: $y'' - y' - 12y = 3te^{2t}$.
4. Determine a suitable form for the particular solution $Y(t)$, if the method of undetermined coefficients is to be used. You do not need to determine the values of the coefficients.

   (i) $y'' + 3y' = 2t^2 + t^2e^{-3t} + \sin(3t)$

(ii) $y'' + y = t(1 + \sin t)$

(iii) $y'' - 5y' + 6y = e^t \cos(2t) + (3t + 4)e^{2t} \sin(t)$

(iv) $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t} \cos(t) + 4t^2 e^{-t} \sin(t)$

(v) $y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t \sin(2t)$

V. Mass-Spring system

1. Solve for the position (in meters) of a mass attached to a spring with **no** damping if the mass is $m =$ 1kg, the spring constant is $k = 4\frac{N}{m}$, and $x(0) = -3$m and $x'(0) = 6\frac{m}{s}$. Also write your answer in $A\cos(\omega t - \alpha)$ form.

2. Solve for the position (in meters) of a mass attached to a spring with damping if the mass is $m = 3$kg, the damping constant is $c = 2\frac{N \cdot s}{m}$, the spring constant is $k = \frac{37}{3}\frac{N}{m}$, and $x(0) = 3$m and $x'(0) = 6\frac{m}{s}$. Also write your answer in $Ae^{-\rho}\cos(\omega t - \alpha)$ form.

3. A mass of 0.5 kilograms stretches a spring 0.14 meters. Suppose the mass is displaced an additional 0.06m in the positive (downward) direction and then released with an initial upward velocity of 8m/s The mass is in a medium that exerts a viscous resistance of 12N when the mass has a velocity of 6m/s. Write an IVP for the position x (in meters) of the mass at any time t (in seconds). Use $g = 9.8$ m/s^2 for the acceleration due to gravity.

4. For the following, choose the best description of the system from the following:
   Simple Harmonic Motion (SHM) Overdamped (OD) Underdamped (UD) Critically Damped (CD) Beating (B) Resonant (R) Steady-State plus Transient (SST)

   a. $x'' + 4x = 0$

   b. $2x'' + 7x' + 3x = 0$

   c. $y'' + (1.8)^2 y = \cos(2t)$

   d. $y'' + 4y = \cos(2t)$

   e. $x'' + x' + x = 0$

   f. $y'' + y' + y = \cos(t)$

   g. $x'' + 2x' + x = 0$

5. The motion of a force mass-spring system is described by the following IVP:

$$x'' + 9x = \cos(3t), \qquad x(0) = 0, \qquad x'(0) = 0$$

Explain why you expect resonance to occur.

6. Solve for the motion of a force mass-spring system is described by the following IVP:

$$x'' + (2.8)^2 x = \cos(3t), \qquad x(0) = 0, \qquad x'(0) = 0$$

7. A mass $m = 1\ kg$ is attached to a spring with constant $k = 2$ N/m and damping constant γ Ns/m. Determine the value of γ so that the motion is critically damped.

VI. Laplace Transform

1. Use the definition of the Laplace transform to find $F(s) = \mathcal{L}\{f(t)\}$ for the following functions.

   (a) $f(t) = \begin{cases} 3, & 0 \le t < 4 \\ 0, & 4 \le t < \infty \end{cases}$

   (b) $f(t) = \begin{cases} 0, & t < 2 \\ 6, & 2 \le t \end{cases}$

   (c) $f(t) = \begin{cases} 0, & t < 2 \\ 5e^{-3t}, & 2 \le t \end{cases}$

   (d) $f(t) = \begin{cases} 2e^{t}, & t < 1 \\ 2e, & 1 \le t \end{cases}$

Given table:

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
	$y(t)$	$Y(s)$
1	1	$\frac{1}{s}$
2	t^n	$\frac{n!}{s^{n+1}}$
3	e^{at}	$\frac{1}{s-a}$
4	$\cos(bt)$	$\frac{s}{s^2+b^2}$
5	$\sin(bt)$	$\frac{b}{s^2+b^2}$
6	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
7	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
8	$y'(t)$	$sY(s) - y(0)$
9	$y''(t)$	$s^2Y(s) - sy(0) - y'(0)$

2. Find the Laplace transform of the following functions.

   (a) $f(t) = 6e^{-2t}\sin(3t)$

   (b) $f(t) = 6t^3 + 5e^{3t}$

   (c) $f(t) = 4\cos(2t) - 2\sin(2t)$

3. Find the inverse Laplace transform:

   (a) $F(s) = \frac{7}{s^3} + \frac{9}{s-4}$

   (b) $F(s) = \frac{8}{s^2-s-6}$

   (c) $F(s) = \frac{7s+2}{s^2+9}$

   (d) $F(s) = \frac{3s+8}{s^2+4s+29}$

4. The transform of the solution to a certain differential equation is given by $Y(s) = \frac{2s-7}{s^2+10}$. Determine the solution $y(t)$ of the differential equation.

5. Suppose that the function $y(t)$ satisfies the DE $y'' - 2y' - y = 3\sin(4t)$, with initial values $y(0) = -1$, $y'(0) = 1$. Find the Laplace transform of $y(t)$ (and solve for Y(s)).

6. Consider the following IVP: $y'' + 6y' + 13y = 2t^3$, $y(0) = 2$, $y'(0) = -1$. Find the Laplace transform of the solution $y(t)$ (and solve for Y(s))

7. Consider the following IVP: $y'' + 81y = 0$, $y(0) = 2$, $y'(0) = -4$.
   (a) Find the Laplace transform of the solution $y(t)$ (and solve for Y(s)).
   (b) Invert the transform to solve for $y(t)$.

8. Consider the following IVP: $y'' + 3y' = 0$, $y(0) = -2$, $y'(0) = 3$.
   (a) Find the Laplace transform of the solution $y(t)$ (and solve for Y(s)).
   (b) Invert the transform to solve for $y(t)$.

9. Consider the following IVP: $y'' + 6y' + 13y = 0$, $y(0) = 2$, $y'(0) = -1$.
   (a) Find the Laplace transform of the solution $y(t)$ (and solve for Y(s)).
   (b) Invert the transform to solve for $y(t)$.

10. Consider the following IVP: $y'' - 3y' - 10y = 5$, $y(0) = 2$, $y'(0) = -4$.
   (a) Find the Laplace transform of the solution $y(t)$.
   (b) Find the solution $y(t)$ by inverting the transform.

ANSWERS TO TEST 2 PRACTICE PROBLEMS

I

1. a) $4 < t$ or in interval form $(4, \infty)$ b) $-1 < t < \frac{\pi}{2}$ or $\left(-1, \frac{\pi}{2}\right)$ (c) $0 < t < 4$ or $(0, 4)$

2. $3e^{4t}$

3. (i) $t > 0$

(iii) $W(y_1, y_2) = \frac{1}{t^3}$. Since the Wronskian is nonzero on I, $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions.

(iv) $y = \frac{1+2\ln(t)}{t}$

4. By the principle of superposition, B, C, D

5. C: te^t is not a solution; D: the two functions are not linearly independent

II.

1. (a) $y = -e^{3t}\sin(2t) + e^{3t}\cos(2t)$

(b) $y(t) = e^{-2t} - 2te^{-2t}$

(c) $y(t) = 3e^{-2t/3} + 4e^{-t/2}$

III.

1. $y(t) = \frac{c_1}{t} + c_2\frac{\ln(t)}{t}$ **2.** $y(t) = c_1 t^3 + c_2\sqrt{t}$

IV.

1. $y_p = 3t^2 - 12t + 18 + \frac{5}{9}e^{2t}$ **;** general: $y = c_1e^{-t} + c_2te^{-t} + 3t^2 - 12t + 18 + \frac{5}{9}e^{2t}$

2. $y_p = \frac{6}{65}\cos(3t) - \frac{22}{65}\sin(3t)$ **;** general: $y = c_1e^{-t} + c_2e^{2t} + \frac{6}{65}\cos(3t) - \frac{22}{65}\sin(3t)$

3. $y_p = \left(-\frac{3}{10}t - \frac{9}{100}\right)e^{2t}$ **;** general: $y = c_1e^{-3t} + c_2e^{4t} + \left(-\frac{3}{10}t - \frac{9}{100}\right)e^{2t}$

4.

(i) $Y(t) = t(At^2 + Bt + C) + t(Dt^2 + Et + F)e^{-3t} + G\sin(3t) + H\cos(3t)$

(ii) $Y(t) = At + B + t(Ct + D)\sin(t) + t(Et + F)\cos(t)$

(iii) $Y(t) = Ae^t\cos(2t) + Be^t\sin(2t) + (Ct + D)e^{2t}\cos(t) + (Et + F)e^{2t}\sin(t)$

(iv) $Y(t) = Ae^{-t} + t(Bt^2 + Ct + D)e^{-t}\cos(t) + t(Et^2 + Ft + G)e^{-t}\sin(t)$

(v) $Y(t) = At^2 + Bt + C + t^2(Dt + E)e^{2t} + (Ft + G)\cos(2t) + (Ht + I)\sin(2t)$

V.

1. $x(t) = -3\cos(2t) + 3\sin(2t) = \sqrt{18}\cos\left(2t - \frac{3\pi}{4}\right)$

2. $x(t) = 3e^{-t/3}\cos(2t) + \frac{7}{2}e^{-t/3}\sin(2t) = \frac{\sqrt{85}}{2}e^{-t/3}\cos\left(2t - \tan^{-1}\left(\frac{7}{6}\right)\right)$

3. $0.5x'' + 2x' + 35x = 0,\ x(0) = 0.06,\ x'(0) = -8$

4.

a. SHM (Undamped)
b. OD
c. B
d. R
e. Underdamped
f. SST
g. CD

5. (a) $\omega_0 = 3 = \omega$

6. (a) $\omega_0 = 2.8 \approx 3 = \omega$ (b) $x(t) = \frac{1}{1.16}(\cos(2.8t) - \cos(3t)) = \frac{25}{29}(\cos(2.8t) - \cos(3t))$

7. $2\sqrt{2}$

VI.

1. (a) $\frac{3-3e^{-4s}}{s}$ (b) $\frac{6e^{-2s}}{s}$ (c) $\frac{5e^{-2(s+3)}}{s+3}$ (d) $\frac{2s-2e^{-(s-1)}}{(s-1)s}$

2. (a) $\frac{18}{(s+2)^2+9}$ (b) $\frac{36}{s^4} + \frac{5}{s-3}$ (c) $\frac{4s-4}{s^2+4}$

3. (a) $\frac{7}{2}t^2 + 9e^{4t}$ (b) $-\frac{8}{5}e^{-2t} + \frac{8}{5}e^{3t}$

(c) $7\cos(3t) + \frac{2}{3}\sin(3t)$ (d) $3e^{-2t}\cos(5t) + \frac{2}{5}e^{-2t}\sin(5t)$

4. $y(t) = 2\cos(\sqrt{10}\,t) - \frac{7}{\sqrt{10}}\sin(\sqrt{10}\,t)$

5. $Y(s) = \frac{-s+3}{s^2-2s-1} + \frac{12}{(s^2-2s-1)(s^2+16)}$

6. (a) $Y(s) = \frac{2s+11}{s^2+6s+13} + \frac{12}{s^4(s^2+6s+13)}$

7. (a) $Y(s) = \frac{2s-4}{s^2+81}$ (b) $y(t) = 2\cos(9t) - \frac{4}{9}\sin(9t)$

8. (a) $Y(s) = \frac{-2s-3}{s^2+3s}$ (b) $y(t) = -1 - e^{-3t}$

9. (a) $Y(s) = \frac{2s+11}{s^2+6s+13}$ (b) $y(t) = 2e^{-3t}\cos(2t) + \frac{5}{2}e^{-3t}\sin(2t)$

10. (a) $Y(s) = \frac{2s-10}{s^2-3s-10} + \frac{5}{s(s^2-3s-10)} = \frac{2s^2-10s+5}{s(s-5)(s+2)}$

(b) $y(t) = -\frac{1}{2} + \frac{1}{7}e^{5t} + \frac{33}{14}e^{-2t}$