LIST OF CONCEPTS AND SKILLS FOR CHAPTER 6-7

New material covers sections 6.3-6.5, 7.1-7.6, 7.8 from MAT 275-Elementary Differential Equations, by Boyce and DePrima - WileyPLUS, 11th edition. (Material from earlier sections are on previous reviews.)

CHAPTER 6: THE LAPLACE TRANSFORM

6.3: Step Functions

- Know the definition of the unit step function (Heaviside function), \( u_c(t) = \text{step}(t - c) \), and how to write a piecewise function in terms of the unit step functions and use the appropriate entry in the table to find the Laplace transform.
- Know how to graph functions involving the unit step function.
- Know how to apply Theorem 6.3.1: \( \mathcal{L}\{u_c(t)f(t - c)\} = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs}F(s) \), and conversely, \( \mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t - c) \), where \( F(s) = \mathcal{L}\{f(t)\} \) and \( f(t) = \mathcal{L}^{-1}\{F(s)\} \).
- Know how to use the translation property: \( \mathcal{L}\{e^{ct}f(t)\} = F(s-c) \).

6.4: Differential Equations with Discontinuous Forcing Equations.

Know how to solve Differential Equations where the forcing term is given by a piecewise continuous function. In these cases the function needs to be written in terms of unit step functions \( u_c(t) \) in order to evaluate the Laplace.

6.5: Impulse Functions

Know the definition of the Dirac delta function, \( \delta(t - t_0) \), and know how to solve differential equations where the forcing terms involves delta functions.

Some Laplace transform formulas:

<table>
<thead>
<tr>
<th>( f(t) = \mathcal{L}^{-1}{F(s)} )</th>
<th>( F(s) = \mathcal{L}{f(t)} )</th>
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<tbody>
<tr>
<td>( f(t) = \mathcal{L}^{-1}{F(s)} )</td>
<td>( F(s) = \mathcal{L}{f(t)} )</td>
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<tr>
<td>( y(t) )</td>
<td>( Y(s) )</td>
</tr>
<tr>
<td>1</td>
<td>( 1 )</td>
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<tr>
<td>2</td>
<td>( t^n )</td>
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<tr>
<td>3</td>
<td>( e^{at} )</td>
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<tr>
<td>4</td>
<td>( \cos(bt) )</td>
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<tr>
<td>5</td>
<td>( \sin(bt) )</td>
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<tr>
<td>6</td>
<td>( e^{at} \cos(bt) )</td>
</tr>
<tr>
<td>7</td>
<td>( e^{at} \sin(bt) )</td>
</tr>
<tr>
<td>8</td>
<td>( u_c(t) )</td>
</tr>
<tr>
<td>9</td>
<td>( u_c(t)f(t) )</td>
</tr>
<tr>
<td>10</td>
<td>( u_c(t)f(t - c) )</td>
</tr>
<tr>
<td>11</td>
<td>( \delta(t - c) )</td>
</tr>
<tr>
<td>12</td>
<td>( y'(t) )</td>
</tr>
<tr>
<td>13</td>
<td>( y''(t) )</td>
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</tbody>
</table>
CHAPTER 7: LINEAR SYSTEMS

Section 7.1: Introduction to linear systems
- Know how to transform a linear ODE of any order $n$ into a system of $n$ first-order ODEs.
- Know how to write a system of first-order linear ODEs in matrix form.
- Know how to solve simple two-dimensional systems by rewriting them as a single higher order ODE.
- Know how to determine the trajectories of the phase portraits for simple 2-dimensional systems.
- Set up the system of differential equations and initial conditions that model the mixing of salt water with two interconnected tanks.

Section 7.3: Linear Independence, Eigenvalues and Eigenvectors
- Know the definition of linearly independent and linearly dependent vectors.
- Know how to find eigenvalues and eigenvectors of a matrix.

Section 7.4: Basic Theory of Systems of First Order Linear Equations
- Know how to write a linear system in the form $x' = P(t)x + f(t)$ with $P(t)$ the coefficient matrix and $x$ and $f(t)$ the appropriate vectors.
- Know how to verify (by substituting) that a given vector function is a solution to a given system.
- Given an $n \times n$ homogeneous linear system of differential equations written in matrix form as $x' = P(t)x$ with $P(t)$ a matrix whose entries are continuous, know how to find the Wronskian of $n$ solutions $x_1(t), x_2(t), ..., x_n(t)$. If the Wronskian of the $n$ solutions $W(x_1(t), x_2(t), ..., x_n(t))$ is nonzero at every point of some interval $I$, then the solutions are linearly independent on $I$.
- Given the general solution, know how to use the Initial Conditions to determine the value of the constants $c_1, c_2, ..., c_n$.

Section 7.5: Homogeneous Linear Systems with Constant coefficients
Consider a system of $n$ linear homogeneous equations with constant coefficients $x' = Ax$ where the coefficient matrix $A$ is real valued.
If all the eigenvalues of $A$ are real and distinct, then there are $n$ linearly independent eigenvectors and the general solution of the system is given by $x = c_1v_1e^{r_1t} + c_2v_2e^{r_2t} + ... + c_nv_ne^{r_nt}$ where $r_1, r_2, ..., r_n$ are the eigenvalues and $v_1, v_2, ..., v_n$ are the corresponding $n$ linearly independent eigenvectors.

Section 7.6: Complex Eigenvalues
Consider a system of $n$ linear homogeneous equations with constant coefficients $x' = Ax$ where the coefficient matrix $A$ is real valued.
If the eigenvalues are complex, then they occur in conjugate pairs and so do the associated eigenvectors. Let $r = p + qi$ be a complex eigenvalue and $v = a + ib$ be the associated eigenvector, then, using Euler's formula, the solution can be written as
$$x(t) = ve^{rt} = (a + ib)e^{pt}(\cos(qt) + i\sin(qt))$$
$$= e^{pt}(a\cos(qt) - b\sin(qt)) + ie^{pt}(a\sin(qt) + b\cos(qt))$$
Taking the real and imaginary part of $x(t)$ we find the **two real valued** solutions of the system:

$$x_1(t) = e^{pt}(a \cos(qt) - b \sin(qt)) \quad \text{and} \quad x_2(t) = e^{pt}(a \sin(qt) + b \cos(qt))$$

It can be shown that $x_1(t)$ and $x_2(t)$ are linearly independent and therefore they can be used to write the general solution $x = c_1x_1 + c_2x_2$.

**NOTE:** Rather than memorizing the formulas for $x_1(t)$ and $x_2(t)$, you should be able to derive them for every specific example using Euler's formula and separating the real and imaginary parts.

**Section 7.8: Repeated Eigenvalues**

Consider a system of two linear homogeneous equations with constant coefficients $x' = Ax$ where the coefficient matrix $A$ is real valued.

If the eigenvalues are repeated with eigenvalue $r$ (real and multiplicity two), then for any eigenvector, $v_1$, for $r$, we have that $x_1(t) = v_1e^{rt}$ is a solution to $x' = Ax$.

A second linearly independent solution is of the form $x_2(t) = (v_2 + tv_1)e^{rt}$ where $(A - rl)v_2 = v_1$. (the vector $v_2$ is not be unique). The general solution is $x = c_1x_1 + c_2x_2$. 