LIST OF CONCEPTS AND SKILLS FOR CHAPTER 6-7

New material covers sections 6.3-6.5, 7.1-7.6, 7.8 from *MAT 275-Elementary Differential Equations*, by Boyce and DePrima - WileyPLUS, 11th edition. (Material from earlier sections are on previous reviews.)

CHAPTER 6: THE LAPLACE TRANSFORM

6.3: Step Functions

- Know the definition of the unit step function (Heaviside function), $u_c(t) = step(t-c)$, and how to write a piecewise function in terms of the unit step functions and use the appropriate entry in the table to find the Laplace transform.
- Know how to graph functions involving the unit step function.
- Know how to apply Theorem 6.3.1: $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\,\mathcal{L}\{f(t)\} = e^{-cs}F(s)$, and conversely,

  $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$, where $F(s) = \mathcal{L}\{f(t)\}$ and $f(t) = \mathcal{L}^{-1}\{F(s)\}$.
- Know how to use the translation property: $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$.

6.4: Differential Equations with Discontinuous Forcing Equations.

Know how to solve Differential Equations where the forcing term is given by a piecewise continuous function. In these cases the function needs to be written in terms of unit step functions $u_c(t)$ in order to evaluate the Laplace.

6.5: Impulse Functions

Know the definition of the Dirac delta function, $\delta(t-t_0)$, and know how to solve differential equations where the forcing terms involves delta functions.

Some Laplace transform formulas:

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
	$y(t)$	$Y(s)$
1	1	$\frac{1}{s}$
2	t^n	$\frac{n!}{s^{n+1}}$
3	e^{at}	$\frac{1}{s-a}$
4	$\cos(bt)$	$\frac{s}{s^2+b^2}$
5	$\sin(bt)$	$\frac{b}{s^2+b^2}$
6	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
7	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
8	$u_c(t)$	$\frac{e^{-cs}}{s}$
9	$u_c(t)f(t)$	$e^{-cs}\mathcal{L}\{f(t+c)\}$
10	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
11	$\delta(t-c)$	e^{-cs}
12	$y'(t)$	$sY(s) - y(0)$
13	$y''(t)$	$s^2Y(s) - sy(0) - y'(0)$

CHAPTER 7: LINEAR SYSTEMS

Section 7.1: Introduction to linear systems

- Know how to transform a linear ODE of any order n into a system of n first-order ODEs.
- Know how to write a system of first-order linear ODEs in matrix form.
- Know how to solve simple two-dimensional systems by rewriting them as a single higher order ODE.
- Know how to determine the trajectories of the phase portraits for simple 2-dimensional systems.
- Set up the system of differential equations and initial conditions that model the mixing of salt water with two interconnected tanks.

Section 7.3: Linear Independence, Eigenvalues and Eigenvectors

- Know the definition of linearly independent and linearly dependent vectors.
- Know how to find eigenvalues and eigenvectors of a matrix.

Section 7.4: Basic Theory of Systems of First Order Linear Equations

- Know how to write a linear system in the form $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{f}(t)$ with $P(t)$ the coefficient matrix and $\mathbf{x}$ and $\mathbf{f}(t)$ the appropriate vectors.
- Know how to verify (by substituting) that a given vector function is a solution to a given system.
- Given an $n \times n$ homogeneous linear system of differential equations written in matrix form as $\mathbf{x}' = P(t)\mathbf{x}$ with $P(t)$ a matrix whose entries are continuous, know how to find the Wronskian of n solutions $\mathbf{x_1}(t),\ \mathbf{x_2}(t), \dots, \mathbf{x_n}(t)$. If the Wronskian of the n solutions $W(\mathbf{x_1}(t),\ \mathbf{x_2}(t), \dots, \mathbf{x_n}(t))$ is nonzero at every point of some interval I, then the solutions are linearly independent on I.
- Given the general solution, know how to use the Initial Conditions to determine the value of the constants $c_1, c_2, \dots, c_n$.

Section 7.5: Homogeneous Linear Systems with Constant coefficients

Consider a system of n linear homogeneous equations with constant coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where the coefficient matrix $\mathbf{A}$ is real valued.

If all the <u>eigenvalues</u> of $\mathbf{A}$ are <u>real</u> and <u>distinct</u>, then there are n linearly independent eigenvectors and the general solution of the system is given by $\mathbf{x} = c_1\mathbf{v}_1 e^{r_1 t} + c_2\mathbf{v}_2 e^{r_2 t} + \cdots + c_n\mathbf{v}_n e^{r_n t}$ where $r_1, r_2, \dots, r_n$ are the eigenvalues and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are the corresponding n linearly independent eigenvectors.

Section 7.6: Complex Eigenvalues

Consider a system of n linear homogeneous equations with constant coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where the coefficient matrix $\mathbf{A}$ is real valued.

If the <u>eigenvalues</u> are <u>complex</u>, then they occur in conjugate pairs and so do the associated eigenvectors. Let $r = p + qi$ be a complex eigenvalue and $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ be the associated eigenvector, then, using Euler's formula, the solution can be written as

$$\mathbf{x}(t) = \mathbf{v}e^{rt} = (\mathbf{a} + i\mathbf{b})e^{pt}(\cos(qt) + i\sin(qt))$$
$$= e^{pt}(\mathbf{a}\cos(qt) - \mathbf{b}\sin(qt)) + ie^{pt}(\mathbf{a}\sin(qt) + \mathbf{b}\cos(qt))$$

Taking the real and imaginary part of $\mathbf{x}(t)$ we find the **two real valued** solutions of the system:

$$\mathbf{x_1}(t) = e^{pt}(\mathbf{a}\cos(qt) - \mathbf{b}\sin(qt)) \quad \text{and} \quad \mathbf{x_2}(t) = e^{pt}(\mathbf{a}\sin(qt) + \mathbf{b}\cos(qt))$$

It can be shown that $\mathbf{x_1}(t)$ and $\mathbf{x_2}(t)$ are linearly independent and therefore they can be used to write the general solution $\boldsymbol{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2}$.

NOTE: Rather than memorizing the formulas for $\mathbf{x}_1$(t) and $\mathbf{x}_2$(t), you should be able to derive them for every specific example using Euler's formula and separating the real and imaginary parts.

Section 7.8: Repeated Eigenvalues

Consider a system of two linear homogeneous equations with constant coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where the coefficient matrix $\mathbf{A}$ is real valued.

If the <u>eigenvalues</u> are <u>repeated</u> with eigenvalue r (real and multiplicity two), then for any eigenvector, $\mathbf{v_1}$, for r, we have that $\mathbf{x_1}(t) = \mathbf{v_1}e^{rt}$ is a solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

A second linearly independent solution is of the form $\mathbf{x_2}(t) = (\mathbf{v_2} + \mathrm{t}\mathbf{v_1})e^{rt}$ where $(\mathbf{A} - r\mathbf{I})\mathbf{v_2} = \mathbf{v_1}$. (the vector $\mathbf{v_2}$ is not be unique). The general solution is $\boldsymbol{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2}$.