Answers: MAT 210 Final Exam Review Questions Combined

Limits (sections 10.1, 10.3)

1. a. 22

   b. $\frac{6}{13}$

   c. 3

2.

   a. $\lim_{x\to\infty}\frac{3\,x^3-3}{-13x^3-4x^2-2} = \lim_{x\to\infty}\frac{3-\frac{3}{x^3}}{-13-\frac{4}{x}-\frac{2}{x^3}} = \frac{3}{-13} = -\frac{3}{13}$, as $x \to \infty$ the read terms aproaches 0.

   As $x \to \infty$, the numerator approaches ∞ and the denominator approaches $-\infty$. This limit expression has a $\frac{\infty}{-\infty}$ **type of indeterminate form (**Again this is not a number and not the answer.) Divide each term of the rational function by x^n, where n is the largest power of x in the expression. In this problem divide by x^3, where 3 is the largest power of x. Take the limit as $x \to \infty$, each $\frac{non-zero\ \#}{x^n}$ term approaches 0. Thus, for large x values (as $x \to \infty$ or as $x \to -\infty$), all terms of a rational function are negligible except the leading terms since the leading terms in both, in the numerator and denominator much larger compared to all the other terms. Therefore $\lim_{x\to\infty}\frac{3\,x^3-3}{-13x^3-4x^2-2} = \lim_{x\to\infty}\frac{3x^3}{-13x^3} = -\frac{3}{13}$. In general, if the degree of the numerator of a rational function is the same as the degree of the denominator then the limit will be the ratio of the leading terms as $x \to \infty$ or as $x \to -\infty$.

   b. $\lim_{x\to\infty}\frac{7\,x^2-5}{19x^3-3x-7} = \lim_{x\to\infty}\frac{7x^2}{19x^3} = \lim_{x\to\infty}\frac{7}{19x} = 0$.

   As $x \to \infty$, the numerator approaches ∞ and the denominator approaches ∞. This limit expression has a $\frac{\infty}{\infty}$ **type of indeterminate form.** Again this is not a number and not the answer. In general, if the degree of the numerator of a rational function is smaller than the degree of the denominator then the limit is 0 as $x \to \infty$ or as $x \to -\infty$.

   c. $\lim_{x\to\infty}\frac{-9\,x^5+5}{12x^3-3x-7} = \lim_{x\to\infty}\frac{-9x^5}{12x^3} = \lim_{x\to\infty}\frac{-9x^2}{12} = -\infty$.

   As $x \to \infty$, the numerator approaches $-\infty$ and the denominator approaches ∞. This limit expression has a $\frac{-\infty}{\infty}$ **type of indeterminate form.** In general, if the degree of the numerator of a rational function is greater than the degree of the denominator then the limit does not exist. The limit is either ∞ or $-\infty$ depending on the sign of the leading coefficients.

   All the limit expressions discussed above have $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$ indeterminate form, so there is no general rule to find the limit, each individual problem requires further analysis to determine the limit.

d. $\lim_{x\to\infty} \frac{-3}{11x^3} = 0$

As $x \to \infty$, the denominator approaches ∞. This limit expression has a $\frac{\boldsymbol{non-zero\,\#}}{\infty}$ **type of determinate form,** thus the limit is 0. A non-zero number divided by a large number results a number a number that is close to 0. As the dividend increases without bound the value of the fraction is getting arbitrarily close to 0.

e. $\lim_{x\to\infty} \frac{-3x^3}{11} = -\infty$

As $x \to \infty$, the numerator approaches $-\infty$. This limit expression has a $\frac{-\infty}{\boldsymbol{non-zero\,\#}}$ **type of determinate form,** thus the limit is $-\infty$.

3. 2

4. 0

5. 2

6. ∞

7. 0

8. $\lim_{t\to+\infty} m(t) = 0.$ In the long term, the amount of drug in the blood will completely disappear.

9. $\lim_{t\to+\infty} p(t) = 500.$ The population of squirrels will approach 500 as the time increases (in the long run).

10.

a) $\lim_{t\to+\infty} W(t) = \infty.$ In the long term, the popularity of Twitter among social media sites will increase without bound.

b) $\lim_{t\to+\infty} \frac{W(t)}{L(t)} = 8.25.$ In the long term, the popularity of Twitter will increase by 8.25 more than LinkedIn.

Note: A percentage can't rise beyond 100, so extrapolating the models to obtain long term predictions gives meaningless results.

Rates of Change (sections 10.4, 10.5)

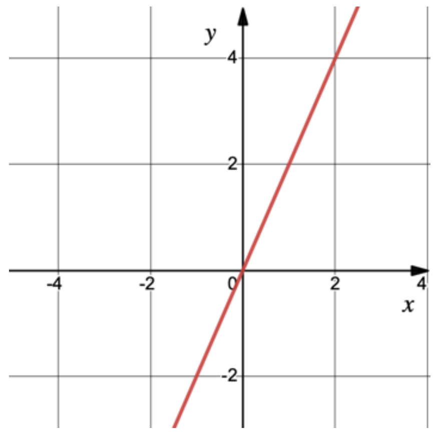
1. 21
2. - 2 dollars/day
3. 0.3125 million barrels per year.

U.S. daily oil imports from a certain country increased by an average rate of 0.3125 million barrels per year over the period 1991 to 1999.

4.

a) The sign of f' is negative on the interval $(-\infty, 0)$, it is 0 at $x = 0$, then positive on the interval $(0, \infty)$

b) f' is always increasing



5.

a) 425/9 beetles per week

b)

A) F
B) T
C) T
D) F
E) T
F) F
G) T
H) F
I) T

6.

A. In 1998 the number of students who took the AP Calculus exam is 156,682 and this number is increasing at a rate of 9,644 student per year.

Differentiation (Power Rule, Product Rule, Quotient Rule, Contant Multiple Rule) **(sections 11.1, 11.3)**

1.

a) $f'(x) = 8x$

b) $f'(x) = -2$

c) $f'(x) = 4x + 6$

d) $f'(x) = \frac{9}{x^5}$

e) $f'(x) = -4x^{-3/2} + 4x^{-2}$

f) $g'(x) = 6x + 8 + 12x^{-7} - 11.5x^{1.3}$

g) $h'(x) = 72x^2 - 32x + 72$

h) $k'(x) = -0.5x^{-2}$

i) $m'(x) = 70x - 21x^{-4} + 49x^{-2}$

j) $v'(x) = \frac{20}{(4x+8)^2}$

k) $p'(x) = \frac{2x\cdot(x^3-2x)-(x^2+1)\cdot(3x^2-2)}{(x^3-2x)^2} = \frac{-x^4-5x^2+2}{(x^3-2x)^2}$

Applications to Derivatives and Rates of Change

Tangent Lines

1. $y = 8x - 6$
2. $y = 6x - 11$
3. $f'(1) = 17$

Average Velocity and Instantaneous Velocity

1.
   a) 2 m/s
   b) 3 m/s

Marginal Analysis (section 11.2)

1.

   a) $MP(x) = 8 - \frac{1}{\sqrt{x}}$

   b) 7.8 dollars per box. After 25 boxes of cookies have been sold, the total profit will increase by about 7.8 dollars per additional box sold, or the profit from selling the 26th box is about 7.8 dollars.

2. $MC(x) = 16x$ $MR(x) = 12x^2 + 2$ $MP(x) = 12x^2 - 16x + 2$
3. profit = 12798 dollars Marginal profit = 506 dollars per book

4.

a) $P(x) = -150 + 6.9x - 0.002x^2$

b) 520 dollars

c) $MP(x) = 6.9 - 0.004x$

d) 6.5 dollars per candle. The profit from selling the 101st candle is about 6.5 dollars. Or the total profit will increase by 6.5 dollars per candle sold, after 100 candles are sold.

Chain Rule (section 11.4)

1.

a) $2(3x^2 + 2x - 8)^4(6x + 2)$

b) $\frac{1}{2}(x^3 - 50x)^{-\frac{1}{2}} \cdot (3x^2 - 50) = \frac{3x^2-50}{2\sqrt{x^3-50x}}$

c) $\frac{4}{3}(6x + 1)^{1/3} \cdot 6 = 8(6x + 1)^{\frac{1}{3}} = 8\sqrt[3]{6x + 1}$

d) $25(-4)(x^2 + x + 2)^{-5} \cdot (2x + 1) = -100(2x + 1)(x^2 + x + 2)^{-5} = -\frac{100(2x+1)}{(x^2+x+2)^5}$

Derivative of Logarithmic and Exponential Functions (section 11.5)

1.

(a) $9 \cdot \frac{1}{2x} \cdot 2 = \frac{9}{x}$

(b) $\frac{15x^2+2x}{5x^3+x^2+4}$

(c) $\frac{3x^2-8}{x^3-8x}$

(d) $1 - \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) = -\ln x$

(e) $4e^{x^5-3x} \cdot (5x^4 - 3)$

(f) $(2x - 2)e^{2x+3} + (x^2 - 2x)e^{2x+3} \cdot 2 = (2x^2 - 2x - 2)e^{2x+3}$

(g) $(-5x^{-2})e^{5/x}$

(h) $-16e^{-2x}$

(i) $5(9x^2 + 2)e^{3x^3+2x}$

Implicit Differentiation (section 11.6)

1.

(a) $\frac{dy}{dx} = \frac{3x^2}{3y^2-1}$

(b) $\frac{dy}{dx} = \frac{15-12xy}{6x^2-2y}$

(c) $\frac{dy}{dx} = \frac{e^x-e^y}{xe^y}$

Maxima and Minima (section 12.1)

1.
   (a) 0
   (b) $-1, 1$
   (c) $-3, 3$
   (d) $-1, 0, 1$
   (e) $(-3, -1)$ and $(0, 1)$
   (f) $(-1, 0)$ and $(1, 3)$

2.
   (a) $-2, 0, 2$
   (b) $-2; 2$

3. Critical point $x = 2$; f has a relative minimum at $x = 2$, which is equal to $f(2) = 2 - 2\ln 2$.

4.
   (a) Critical points: $x = -1, 1$.
   (b) Abs Max = $f(-1) = f(2) = 28$; Abs Min= $f(-3) = -132$.

5.

   (a) Critical point: $x = 2$
   (b) Abs Max = $g(5) = 3^{2/3}$; Abs Min = $g(2) = 0$.

6.
   (a) Critical points: $x = 1, 3$.
   (b) Increasing on $(-\infty, 1) \cup (3, \infty)$; decreasing on $(1, 3)$.
   (c) Abs Max = $k(5) = 60$; Abs Min = $k(-1) = -84$; Rel Max = $k(1) = -4$; Rel Min = $k(3) = -20$.

7.
   (a) Critical points: $x = 0$.
   (b) Increasing on $(0, 3)$; decreasing on $(-2, 0)$.
   (c) Abs Max = $h(3) = e^3 - 3$; Abs Min = $h(0) = 1$; Rel Max = $h(-2) = e^{-2} + 2$; Rel Min: None.

8. **T, T, F, T.**
9.
   (a) No relative extrema
   (b) Relative min
   (c) $(4, \infty)$
   (d) $(-\infty, 4)$

Optimization: Applications to Maximum and Minimum (Section 12.2)

1.
   (a) $P(q) = 68q - 0.1q^2 - (23 + 20q) = -0.1q^2 + 48q - 23$
   (b) Set $P' = -0.2q + 48 = 0$, and solve for q: $q = 240$
   (c) $P(240) = -0.1 \cdot 240^2 + 48 \cdot 240 - 23 = 5737$ dollars.

2. $\bar{C}(x) = 0.02x + 2 + 4000x^{-1}$. Set $\bar{C}'(x) = 0.02 - 4000x^{-2} = 0$ and solve for x: $x = 447$ units.

3. Maximize area $A = xy$ subject to cost $3x + 10y = 120$, where x is the length of the north and south sides, and y is the length of east and west sides.

   Dimension is $x = 20\ ft, y = 6\ ft$. Max area = 120 square feet.

4. Minimize the cost $C = 4x + 8y$ subject to area $xy = 162$, where x is the length of the north and south sides, and y is the length of east and west sides.

   Dimension is $x = 18\ ft, y = 9\ ft$. Minimum cost = 144 dollars.

5. $R(p) = p \cdot q = p(-10p + 4220) = -10p^2 + 4220p$. Set $R'(p) = -20p + 4220 = 0$ and solve for p: $p = \$\ 211$. Maximum revenue is $R(211) = 211 \cdot 2110 = \$445210$.

Higher-order Derivatives, Acceleration and Concavity (Section 12.3)

1.
   (a) $y'' = 8e^{2x-5}$
   (b) $y'' = 14x^{-3} + 5x^{-2}$

2. (a) 0; (b) negative, positive; (c) 0; (d) down, up.
3.
   (a) down, up, down
   (b) $(2, 3)$, $(-\infty, 2) \cup (3, \infty)$;
   (c) $x = 2, 3$
   (d) None
   (e) Abs max at $x = 3.5$, no abs min.

4. $a(t) = s''(t) = -\frac{1}{4}t^{-\frac{3}{2}} + 8$
5. 8 ft/sec^2

Related Rates (Section 12.5)

1. (a) 2827 $\text{cm}^2\text{/sec}$; (b) 479 $\text{cm}^2\text{/sec}$
2. 0.03 cm/sec
3. When $x = 4000$, $\frac{d\bar{C}}{dt} = \frac{d\bar{C}}{dx} \cdot \frac{dx}{dt} = \$ - 2.84$ per player per week. So, the average cost is decreasing at a rate of 2.84 dollars per player per week.

Elasticity (Section 12.6)

1. down, 3.2%, decrease.

2. down, 0.65%, increase.
3.

   (a) $E(p) = -(-18)\cdot\frac{p}{1080-18p} = \frac{18p}{1080-18p}$, so $E(20) = \frac{18\cdot 20}{1080-18\cdot 20} = 0.5$.
   This means the demand will drop by 0.5% for 1% increase from current price \$20.
   (b) $0.5 < 1$, it is inelastic. The price should be raised to increase revenue.
   (c) Solve for the price when $E(p) = 1$. Solving $\frac{18p}{1080-18p} = 1$ gives $p = \$30$.
   (d) $R = pq = 30(1080 - 18\cdot 30) = 16200$ dollars.

4.
   (a) $E(p) = -(-4p+33)\cdot\frac{p}{-2p^2+33p} = \frac{4p-33}{-2p+33}$
   (b) $E(15) = \frac{4(15)-33}{-2(15)+33} = \frac{27}{3} = 9 > 1$. It is elastic. The demand will drop by 9% if the price increases by 1%.
   The price should be lowered from \$15 to increase revenue.

Indefinite Integral (Section 13.1)

1.
   (a) $\frac{2}{5}x^5 + \frac{4}{x} - \frac{5}{4x^4} + 3x + C$
   (b) $7\ln|x| - \frac{1}{18x^6} + C$
   (c) $-\frac{2}{x} - \frac{10}{3}x^{\frac{3}{2}} + C$
   (d) $e^x - \frac{x^{0.7}}{0.7} + C = e^x - \frac{10}{7}x^{0.7} + C$
   (e) $\int x^2 + x - 6\,dx = \frac{x^3}{3} + \frac{x^2}{2} - 6x + C$
   (f) $\int x + 5 - \frac{2}{x}\,dx = \frac{x^2}{2} + 5x - 2\ln|x| + C$

2. $f(x) = 9e^x + 9x - 10$

3. $s(t) = \frac{1}{3}t^3 + 6t - 27$

4.
   $C(x) = 0.25x^2 + \ln|x| + 4.75$ dollars
   $$C(x) = \int C'(x)\,dx = \int 0.5x + \frac{1}{x}\,dx = 0.25x^2 + \ln|x| + K$$
   $C(1) = 5$, So $5 = 0.25\cdot 1^2 + \ln|1| + K$. Then solve for constant K: $K = 5 - 0.25 = 4.75$.

Substitution (Section 13.2)

1.

(a) $16 \cdot \frac{e^{-3x}}{-3} + C = -\frac{16}{3}e^{-3x} + C$

(b) $\frac{(5x-2)^4}{4} \cdot \frac{1}{5} + C = \frac{1}{20}(5x-2)^4 + C$

(c) $\frac{1}{2}\ln|2x-5| + C$

(d) $2e^{x^2-3} + C$

(e) $\frac{1}{22}(x^2+1)^{11} + C$

(f) $-5(-x^2+7)^{\frac{3}{2}} + C$

(g) $\frac{1}{10}(x^3+x-2)^{10} + C$

(h) $\frac{3}{2}(\ln x)^2 + C$

Fundamental Theorem of Calculus; Definite Integral; Left Riemann Sum (Sections 13.3, 13.4)

1.

(a) $\left[6\frac{x^6}{6} + 15\frac{x^5}{5} - 9\frac{x^3}{3} + x\right]_0^1 = [x^6 + 3x^5 - 3x^3 + x]_0^1 = (1+3-3+1) - (0) = 2$

(b) $\left[\frac{x^2}{2} + 5\ln|x|\right]_2^7 = \left(\frac{7^2}{2} + 5\ln 7\right) - \left(\frac{2^2}{2} + 5\ln\ 2\right) = \frac{49}{2} + 5\ln 7 - \frac{4}{2} - 5\ln 2 = \frac{45}{2} + 5\ln\frac{7}{2}$

(c) $\int_1^{10} \frac{1}{x^2}\,dx = \int_1^{10} x^{-2}\,dx = \left[\frac{x^{-1}}{-1}\right]_1^{10} = \left[-\frac{1}{x}\right]_1^{10} = -\frac{1}{10} - (-1) = \frac{9}{10}$

(d) $[-e^{-x+6}]_0^6 = (-e^{-6+6}) - (-e^{-0+6}) = -1 + e^6$

(e) $\left[5\frac{e^{3x}}{3}\right]_{-1}^1 = \left[\frac{5}{3}e^{3x}\right]_{-1}^1 = \frac{5}{3}e^{3(1)} - \frac{5}{3}e^{3(-1)} = \frac{5}{3}(e^3 - e^{-3})$

(f) $[2\ln|x|]_{e^3}^{e^5} = 2\ln(e^5) - 2\ln(e^3) = 2\,(5) - 2(3) = 10 - 6 = 4$

(g) $\left[\frac{e^{2x}}{2}\right]_{\ln 3}^{\ln 5} = \frac{e^{2\ln 5} - e^{2\ln 3}}{2} = \frac{e^{\ln 5^2} - e^{\ln 3^2}}{2} = \frac{25-9}{2} = \frac{16}{2} = 8$

2. $\int_2^b (2x-4)\,dx = 9 >> [x^2 - 4x]_2^b = 9 >> (b^2 - 4b) - (2^2 - 4 * 2) = 9 >> b^2 - 4b + 4 = 9$

$>> b^2 - 4b - 5 = 0 >> (b-5)(b+1) = 0 >> b = 5 \text{ or } b = -1\ (\text{rejected since } b > 0)$. Therefore, b = 5.

3. 22.56

$\Delta x = \frac{b-a}{n} = \frac{3-1}{5} = 0.4,\ x_0 = a = 1, x_1 = x_0 + \Delta x = 1.4, x_2 = 1.8,\ x_3 = 2.2,\ x_4 = 2.6.$

LRS $= \Delta x \cdot \big(f(1) + f(1.4) + f(1.8) + f(2.2) + f(2.6)\big) = 0.4(2 + 5.68 + 10.32 + 15.92 + 22.48) = 22.56$

4. 0.18

$$\Delta x = 0.25, \text{LRS} = 0.25\left(\frac{1}{1+2(2)} + \frac{1}{1+2(2.25)} + \frac{1}{1+2(2.5)} + \frac{1}{1+2(2.75)}\right) = 0.25(0.2 + 0.18 + 0.17 + 0.15) = 0.18$$

Applications of Definite Integrals (Section 13.4)

1. Displacement $= s(6) - s(2) = \int_2^6 v(t)\,dt = \int_2^6 (-t^2 + 8)\,dt = \left(-\frac{t^3}{3} + 8t\right)_2^6$

$$= \left(-\frac{6^3}{3} + 8 * 6\right) - \left(-\frac{2^3}{3} + 8 * 2\right) = (-72 + 48) - \left(-\frac{8}{3} + 16\right) = -24 + \frac{8}{3} - 16 = -\frac{112}{3} \text{ meters}$$

2. Total revenue generated $= R(5000) - R(101) = \int_{101}^{5000} MR\,dx = \int_{101}^{5000} 500e^{-0.001x}\,dx$

$$= \left[\frac{500}{-0.001} e^{-0.001x}\right]_{101}^{5000} = -500\,000\,(e^{-5} - e^{-0.101}) \approx 448597.54 \text{ dollars}$$

3. Total number of hours $= \int_2^5 f(t)\,dt = \int_2^5 (1.1t^2 - 2.6t + 2.3)\,dt = \left(\frac{1.1t^3}{3} - \frac{2.6t^2}{2} + 2.3\,t\right)_2^5$

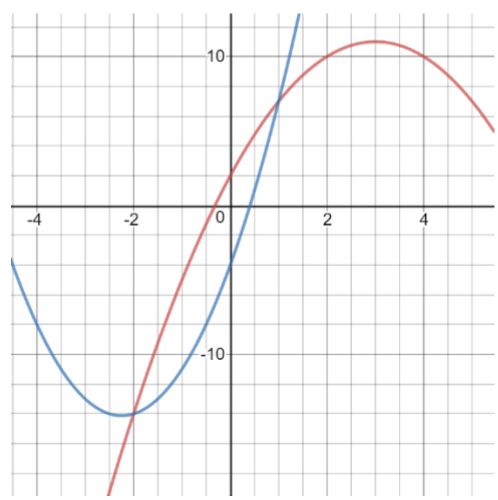
$$= \left(\frac{1.1(5)^3}{3} - \frac{2.6(5)^2}{2} + 2.3\,(5)\right) - \left(\frac{1.1(2)^3}{3} - \frac{2.6(2)^2}{2} + 2.3\,(2)\right) = \frac{45}{2} \text{ million hours of video}$$

4. Area under curve $= \int_0^{16} \sqrt{x}\,dx = \left(\frac{2}{3}x^{\frac{3}{2}}\right)_0^{16} = \left(\frac{2}{3}(16)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(0)^{\frac{3}{2}}\right) = \left(\frac{2}{3}(4^2)^{\frac{3}{2}}\right) = \left(\frac{2}{3}(4^3)\right) = \frac{128}{3}$

Area between Curves (Section 14.2)

1. $\frac{27}{2}$

Details:



Find the intersection points: $-x^2 + 6x + 2 = 2x^2 + 9x - 4$

$$0 = 3x^2 + 3x - 6$$

$$0 = 3(x + 2)(x - 1)$$

So $x = -2$ and $x = 1$.

The area enclosed by the curves from -2 to 1 is

$$\int_{-2}^{1} (\text{top} - \text{bottom})\,dx = \int_{-2}^{1} [(-x^2 + 6x + 2) - (2x^2 + 9x - 4)]\,dx$$

$$= \int_{-2}^{1} (-3x^2 - 3x + 6)\,dx$$

$$= -x^3 - \frac{3}{2}x^2 + 6x|_{-2}^{1}$$

$$= (-1^3 - \frac{3}{2}(1)^2 + 6(1)) - \left(-(-2)^3 - \frac{3}{2}(-2)^2 + 6(-2)\right) = \frac{27}{2}$$

2. $\frac{32}{3}$

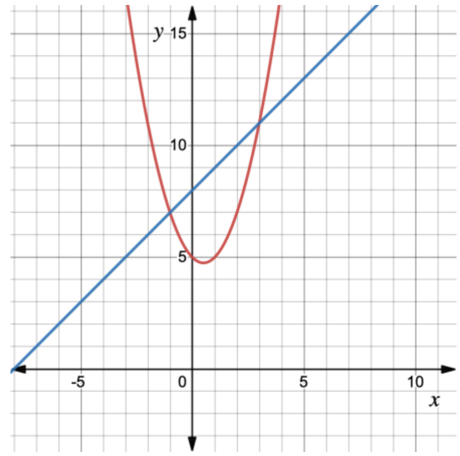
Details:

Find the intersection points: $x^2 - x + 5 = x + 8$

$$0 = x^2 - 2x - 3$$

$$0 = (x+1)(x-3)$$

So $x = 3$ and $x = -1$.



The area enclosed by the curves from -1 to 3 is

$\int_{-1}^{3}(\text{top} - \text{bottom})\,dx = \int_{-1}^{3}[(x+8) - (x^2 - x + 5)]\,dx$

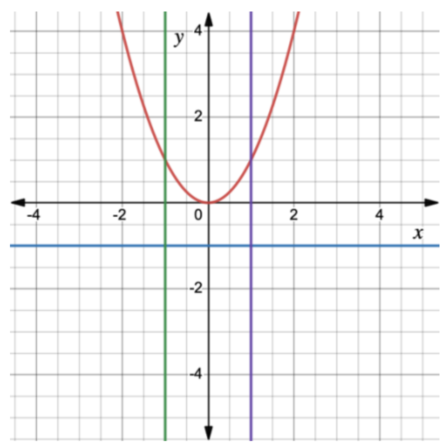
$$= \int_{-1}^{3}(-x^2 + 2x + 3)\,dx$$

$$= \frac{-x^3}{3} + x^2 + 3x|_{-1}^{3}$$

$$= \left(\frac{-3^3}{3} + 3^2 + 3(3)\right) - \left(\frac{-(-1)^3}{3} + (-1)^2 + 3(-1)\right) = 9 - \left(\frac{1}{3} + 1 - 3\right) = \frac{32}{3}$$

3. $\frac{8}{3}$

Details:



The area enclosed by the curves from -1 to 1 is

$\int_{-1}^{1}(\text{top} - \text{bottom})\,dx = \int_{-1}^{1}[x^2 - (-1)]\,dx$

$$= \int_{-1}^{1}(x^2 + 1)\,dx$$

$$= \frac{x^3}{3} + x|_{-1}^{1}$$

$$= \left(\frac{1^3}{3} + 1\right) - \left(\frac{(-1)^3}{3} + (-1)\right) = \frac{4}{3} - \left(-\frac{4}{3}\right) = \frac{8}{3}$$

4. C

Average Value (Section 14.3)

1. $3(e^{1.5} - e^{-0.5})$

The average value of a continuous function $f(x)$ over interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

$$\frac{1}{3-(-1)}\int_{-1}^{3} 6e^{0.5x}\,dx = \frac{1}{4}\cdot 6 \cdot \frac{e^{0.5x}}{0.5}|_{-1}^{3} = 3e^{0.5x}|_{-1}^{3} = 3(e^{1.5} - e^{-0.5})$$

2. 1

The average value of a continuous function $f(x)$ over interval $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

$$\frac{1}{2-(0)}\int_0^2 (x^3-x)\,dx = \frac{1}{2}\left(\frac{x^4}{4}-\frac{x^2}{2}\right)\Big|_0^2 = \frac{1}{2}\left(\frac{2^4}{4}-\frac{2^2}{2}\right) = \frac{1}{2}(4-2) = 1$$

3. 15

The average value of a continuous function $f(x)$ over interval $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

$$\frac{1}{2-(-2)}\int_{-2}^2 (6x^2-4x+7)\,dx = \frac{1}{4}\left(\frac{6x^3}{3}-\frac{4x^2}{2}+7x\right)_{-2}^{2} = \frac{1}{4}(2x^3-2x^2+7x)\Big|_{-2}^{2}$$

$$= \frac{1}{4}[(2(2)^3-2(2)^2+7(2))-(2(-2)^3-2(-2)^2+7(-2))] = \frac{1}{4}\big(22-(-38)\big) = \frac{60}{4}$$

$$= 15$$

Consumer's Surplus and Producer's Surplus (section 14.4)

1.

a. When unit price $p=20$, the demand $q=50$ and when the unit price $p=18.5$, the demand $q=53$. The slope is $\frac{-1.5}{3}=-\frac{1}{2}$ dollar per t-shirt. Thus, $p-20=-\frac{1}{2}(q-50)$, that is $p=-\frac{1}{2}q+45$.

b. When $\bar{p}=15$, then $15=-\frac{1}{2}q+45$, $-30=-\frac{1}{2}q$, $\bar{q}=60$.

$$CS = \int_0^{60}\left(\left(-\frac{1}{2}q+45\right)-15\right)dq = \int_0^{60}\left(-\frac{1}{2}q+30\right)dq = \left(-\frac{1}{4}q^2+30q\right)\Big|_0^{60}$$

$$= -\frac{1}{4}\cdot 60^2 + 30\cdot 60 = -900+1800 = \$\,900.$$

The consumers' surplus is 900 dollars for the first 60 t-shirt sold.

2. When $\bar{p}=160$, then $160=130+e^{0.01q}$, $30=e^{0.01q}$, $\bar{q}=\frac{\ln 30}{0.01}=340.1197382$.

$$PS = \int_0^{\frac{\ln 30}{0.01}}\left(160-(130+e^{0.01q})\right) = \int_0^{\frac{\ln 30}{0.01}}(30-e^{0.01q})dq = \left(30q-\frac{1}{0.01}e^{0.01q}\right)\Big|_0^{\frac{\ln 30}{0.01}}$$

$$= \left(30\cdot\frac{\ln 30}{0.01}-\frac{1}{0.01}\cdot e^{0.01\cdot\frac{\ln 30}{0.01}}\right)-\left(30\cdot 0-\frac{1}{0.01}e^{0.01\cdot 0}\right) = \left(30\cdot\frac{\ln 30}{0.01}-\frac{30}{0.01}\right)+\frac{1}{0.01} = 7303.59.$$

The producers' surplus is \$ 7303.59 for the first 340 items sold.

3.

To find the equilibrium price solve the following system of equations $p=13-q^{\frac{1}{4}}$ and $q=\left(\frac{p-4}{2}\right)^4$ simultaneously, that is find when the supply meets the demand. Substitute $q=\left(\frac{p-4}{2}\right)^4$ for q into $p=13-q^{\frac{1}{4}}$.

Using the identity $\left(a^k\right)^n=a^{k\cdot n}$, we obtain $p=13-q^{\frac{1}{4}}=13-\left(\frac{p-4}{2}\right)^{4\cdot\frac{1}{4}}=13-\left(\frac{p-4}{2}\right)$. Solve the equation

$p = 13 - \left(\frac{p-4}{2}\right)$ for p. Multiplying by 2, we get $2p = 26 - (p - 4) = 30 - p$. Then $3p = 30$ and $\bar{p} = \$\ 10$.

To find the corresponding demand $\bar{q}$, substitute $\bar{\boldsymbol{p}} = \mathbf{10}$ into $q = \left(\frac{p-4}{2}\right)^4$ and solve for q. Thus $\bar{\boldsymbol{q}} = \mathbf{81}$.

The equilibrium price is $\bar{\boldsymbol{p}} = \mathbf{10}$.

$CS = \int_0^{81}((13 - q^{\frac{1}{4}}) - 10)dq = \int_0^{81}(3 - q^{\frac{1}{4}})dq = \left(3q - \frac{1}{1.25}q^{1.25}\right)\Big|_0^{81} = 3 \cdot 81 - \frac{1}{1.25} \cdot 81^{1.25} = \$\ 48.6$

The consumers' surplus for the first 81 items is $\mathbf{48.6}$ dollars.

Before we find the producers' surplus, we need to solve the supply equation for p. The 4th root of both sides of the equation $q = \left(\frac{p-4}{2}\right)^4$ to obtain $q^{\frac{1}{4}} = \left(\frac{p-4}{2}\right)$. Multiplying by 2 and adding 4 to both sides of the equation, we get $p = 2q^{1/4} + 4$. Let us calculate the producer surplus.

$PS = \int_0^{81}\left(10 - \left(2q^{1/4} + 4\right)\right) = \int_0^{81}(6 - 2\,q^{1/4})dq = \left(6q - \frac{2}{1.25}q^{1.25}\right)\Big|_0^{81} = 6 \cdot 81 - \frac{2}{1.25}81^{1.25} = \$\ 97.2$

The producers' surplus for the first 81 items is 97.2 dollars.
The total social gain is the sum of CS and PS, that is $48.6 + 97.2 = 145.8$ dollars for the first 81 items.

4. a) $x = 400$ items

   Details:

$$d(x) = s(x) \gg 200 - 0.2x = 0.3x \gg 200 - 0.5x = 0 \gg x = \frac{200}{0.5} = 400\ items$$

   b) $CS = \$16{,}000$; $PS = \$24{,}000$

   Details:

   The equilibrium price is:

$$\bar{p} = 200 - 0.2\,\bar{x} = 200 - 0.2\,(400) = 200 - 80 = 120$$

   CS:

$$CS = \int_0^{\bar{x}}(d(x) - \bar{p})dx = \int_0^{400}(200 - 0.2x - 120)dx = \int_0^{400}(80 - 0.2x)dx = [80x - \ 0.1\,x^2]_0^{400}$$
$$= 80(400) - 0.1\,(400)^2 = 32{,}000 - 16{,}000 = \$16{,}000$$

   PS:

$$PS = \int_0^{\bar{x}}(\bar{p} - s(x))dx = \int_0^{400}(120 - 0.3x)dx = [120x - \frac{0.3x^2}{2}]_0^{400} = 120(400) - \frac{0.3}{2}\,(400)^2$$
$$= 48{,}000 - 24{,}000 = \$24{,}000$$

5. a) $x = 26$ items

Details:

$$d(x) = s(x) \gg 270.4 - 0.1x^2 = 0.3x^2 \gg 270.4 - 0.4x^2 = 0 \gg x^2 = \frac{270.4}{0.4} = 676 \gg x = \sqrt{676} = 26\ items$$

b) $CS = \$1{,}171.7$; $PS = \$3{,}515.2$

Details:

The equilibrium price is:

$$\bar{p} = 0.3\ \bar{x}^2 = 0.3\ (26)^2 = 202.8$$

CS:

$$CS = \int_0^{\bar{x}} (d(x) - \bar{p})dx = \int_0^{26} (270.4 - 0.1x^2 - 202.8)dx = \int_0^{26} (67.6 - 0.1x^2)dx = [67.6x - \frac{0.1\ x^3}{3}]_0^{26}$$

$$= 67.6(26) - 0.1\ \frac{(26)^3}{3} = \$1{,}171.7$$

PS:

$$PS = \int_0^{\bar{x}} (\bar{p} - s(x))dx = \int_0^{26} (202.8 - 0.3x^2)dx = [202.8x - 0.1x^3]_0^{26} = 202.8(26) - 0.1\ (26)^3 = \$3{,}515.2$$

6. a) $x = 256$ items

Details:

$$d(x) = s(x) \gg \frac{2304}{\sqrt{x}} = 9\sqrt{x} \gg 9x = 2304 \gg x = \frac{2304}{9} = 256\ items$$

b) $CS = \$36{,}864$; $PS = \$12{,}288$

Details:

The equilibrium price is:

$$\bar{p} = 9\sqrt{\bar{x}} = 9\sqrt{256} = 144$$

CS:

$$CS = \int_0^{\bar{x}} (d(x) - \bar{p})dx = \int_0^{256} (\frac{2304}{\sqrt{x}} - 144)dx = [4608\sqrt{x} - 144x]_0^{256} = 4608\sqrt{256} - 144(256)$$

$$= \$36{,}864$$

PS:

$$PS = \int_0^{\bar{x}} (\bar{p} - s(x))dx = \int_0^{256} (144 - 9\sqrt{x})dx = [144x - 9(\frac{2}{3})x^{\frac{3}{2}}]_0^{256} = 144(256) - 6(256)^{\frac{3}{2}} = \$12,288$$

7. $CS = \int_0^{\bar{q}} (D(q) - \bar{p})dq = \int_0^{109.86} (15\mathrm{e}^{-0.01q} - 5)dq = [\frac{15}{-0.01}\mathrm{e}^{-0.01q} - 5q]_0^{109.86} = \left[-1500\,\mathrm{e}^{-0.01(109.86)} - 5(109.86)\right] - \left[-1500\mathrm{e}^{-0.01(0)} - 5(0)\right] = [-1500\,\mathrm{e}^{-1.0986} - 5(109.9)] - [-1500] \approx \$450.49$

8. $PS = \int_0^{\bar{q}} (\bar{p} - S(q))dq = \int_0^{8} (29 - (13 + 2q))dq = \int_0^{8} (16 - 2q)dq = [16q - q^2]_0^8 = (16(8) - 8^2) = \$64$

9. $\bar{q} = 3.5^3$; $PS = \int_0^{\bar{q}} (\bar{p} - S(q))dq = \int_0^{3.5^3} \left(14 - (7 + 2\mathrm{q}^{\frac{1}{3}})\right) dq = \int_0^{3.5^3} \left(7 - 2\mathrm{q}^{\frac{1}{3}}\right) dq = \left[7q - \frac{3}{2} q^{\frac{4}{3}}\right]_0^{3.5^3} =$

$7(3.5^3) - 1.5\,(3.5^3)^{\frac{4}{3}} = 7(3.5^3) - 1.5(3.5)^4 \approx \$75.03$

Improper Integral (Section 14.5)

1.

a) $\int_1^{\infty} \frac{8}{x^2} dx = \lim_{t\to\infty} \int_1^{t} \frac{8}{x^2} dx = \lim_{t\to\infty} ((\frac{-8}{x})\Big|_1^t) = \lim_{t\to\infty} (-\frac{8}{t} + 8) = 8$ since $\lim_{t\to\infty} \left(-\frac{8}{t}\right) = 0.$

Thus, this improper integral converges to 8.

b)

$\int_{-2}^{\infty} e^{-3x}\,dx = \lim_{t\to\infty} \int_{-2}^{\mathrm{t}} e^{-3x}\,dx = \lim_{t\to\infty} -\frac{1}{3} e^{-3x}|_{-2}^{t} = \lim_{t\to\infty} (-\frac{1}{3} e^{-3t} - (-\frac{1}{3} e^{6})) =$

$\lim_{t\to\infty} (-\frac{1}{3} e^{-3t} + \frac{1}{3} e^{6}) = \frac{1}{3} e^{6} = \frac{\mathrm{e}^6}{3}$ since the $\lim_{t\to\infty} \left(-\frac{1}{3} e^{-3t}\right) = 0.$

Thus, this improper integral converges to $\frac{\mathrm{e}^6}{3}$.

c) $\int_1^{\infty} \frac{3}{x} dx = \lim_{t\to\infty} \int_1^{\mathrm{t}} \frac{3}{x} dx = \lim_{t\to\infty} 3\ln|x|\,|_1^t = \lim_{t\to\infty} (3\ln|t| - 3\ln 1) = \infty$.

Since $\lim_{t\to\infty} \ln|t| = \infty$ and $\ln 1 = 0$.

Thus, this improper integral diverges.

d) Converges to 6

e) Converges to $\frac{3}{2}$

f) Diverges to ∞

g) Diverges to ∞

h) Converges to 2.986528

i) Diverges to - ∞

j) Converges to 0.005208

k) Converges to 1

l) Converges to e^2

m) Converges to $\frac{1}{2e^8}$

n) Diverges to ∞

o) Converges to $\frac{1}{2}$