Calculus of Variations

Spring 2023

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Prerequisites: The course will be essentially self-contained. The only prerequisites are familiarity with the Lebesgue integral and the first properties of L^p spaces, as well as a basic knowledge of Sobolev spaces. References will be given for these topics as needed. Regardless of your background, if you are at all interested in this course, please contact me at ddanielli@asu.edu.

Overview: The calculus of variations is one of the classical subjects in mathematics. Its origins can be traced as far back as Queen Dido's isoperimetric problem (∼800 BC): finding which figure, among the ones of fixed perimeter, encloses the largest area. Besides its links with other branches of mathematics, such as geometry and differential equations, it has found applications to classical mechanics, optics, elasticity, economics, image reconstruction, and materials science, just to name a few. Many outstanding mathematicians have contributed to its development, and it still is a very evolving subject. One of the central problems in the calculus of variations is the study of the Dirichlet integral

$$\int_\Omega |\nabla u|^2 dx,$$

motivated by its connection with Laplace's equation $\Delta u = 0$. This problem was also instrumental in the development of several areas of analysis, such as functional analysis, measure theory, distribution theory, and Sobolev spaces. The essence of the calculus of variations is to identify necessary and sufficient conditions that guarantee the existence of minimizers for integral functionals of the type

$$\mathcal{F}(u; \Omega) = \int_\Omega F(x, u, Du)\ dx.$$

In this course we will focus on two – complementary – approaches. On the one hand, we will explore the connection with partial differential equations through the Euler-Lagrange equation. On the other hand, we will present the so-called direct methods, which consist in proving the existence of the minimum of $\mathcal{F}$ directly, rather than resorting to its Euler equation. The central idea is to consider $\mathcal{F}$ as a real-valued mapping on the space of functions taking on $\partial\Omega$ given boundary values, and applying to it a generalization of Weierstrass' theorem on the existence of the minimum of a continuous function. One of the main issues in this approach is to identify the Sobolev spaces as the proper function spaces for compactness results to hold. In turn, the solution of the existence problem in the Sobolev class opens up a series of questions about the regularity of the minimizers. This was a long-standing open problem, until the way to its solution (in a non-direct fashion, since it involves the Euler equation) was opened by the celebrated De Giorgi-Nash-Moser result concerning the Hölder continuity of solutions to uniformly elliptic

PDEs in divergence form with bounded and measurable coefficients. To conclude, we will analyze some examples of paramount importance, such as the obstacle problem, whose study lead to the theory of variational inequalities and free boundary problems.

Textbooks:

- Bernard Dacorogna, *Introduction to the calculus of variations.* Third edition. Imperial College Press, London, 2015. ISBN: 978-1-78326-551-0.
- Enrico Giusti, *Direct methods in the calculus of variations.* World Scientific Publishing Co., Inc., River Edge, NJ, 2003. ISBN: 981-238-043-4.
- David Gilbarg and Neil S. Trudinger, E*lliptic partial differential equations of second order*. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001. ISBN: 3-540-41160-7 35-02
- David Kinderlehrer and Guido Stampacchia, *An introduction to variational inequalities and their applications.* Classics in Applied Mathematics, 31. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. ISBN: 0-89871-466-4.

Required work: One 20-minute presentation during finals week, times TBA. Class attendance is expected.

Approximate schedule (subject to change):

- **Week 1**: Historical perspective and model problems (Dacorogna, Chapter 1)
- **Week 2**: The Euler-Lagrange equation (Dacorogna, Sections 2.1-2.3)
- **Week 3**: The Hamiltonian formulation and the Hamilton-Jacobi equation (Dacorogna, Sections 2.4-2.5)
- **Week 4**: Fields theories and the isoperimetric inequality (Dacorogna, Sections 2.6 and 6.1-6.2)
- **Week 5**: Direct methods: semi-classical theory (Giusti, Sections 1.1-1.2)
- **Week 6**: Barriers; The area functional (Giusti, Sections 1.3-1.4)
- **Week 7**: Convexity (Giusti, Sections 4.1-4.2)
- **Week 8**: Semicontinuity (Giusti, Sections 4.3-4.4)
- **Week 9**: Quasiconvex functionals (Giusti, Sections 5.1-5.2)
- **Week 10**: The quasi-convex envelope; Ekeland's variational principle (Giusti, Sections 5.3-5.4)
- **Week 11**: An existence result (Giusti, Sections 5.5-5.6)
- **Week 12**: The De Giorgi-Nash-Moser theory (Gilbarg & Trudinger, Sections 8.3-8.5)
- **Week 13**: Harnack inequality and Hölder continuity (Gilbarg & Trudinger, Sections 8.7-8.9)

- **Week 14**: Variational inequalities (Kinderleher and Stampacchia, Chapter II, Sections 1-5)
- **Week 15**: The obstacle problem (Kinderleher and Stampacchia, Chapter II, Sections 6-7)

Goals:

- Master the fundamental notions in the calculus of variations.
- Recognize the connections with other areas of mathematics.
- Develop the ability to understand and present research papers in the topic.