MAT 210 Sections 14.4 – 14.5 Review Questions

In addition of the material covered in exam 1, 2, and 3; the final covers consumer/producer surplus (14.4) and improper integral (14.5). Below if some review questions related to these two sections. The final exam is comprehensive, you need to review the material covered in exam 1, 2 and 3 as well.

Consumer's Surplus and Producer's Surplus (section 14.4)

1. Your shop can sell 50 "I love calculus" t-shirt at \$20 each per day. You decided to drop the price \$1.50 per shirt and this results in 3 more t-shirts sold per day.

a. Write a linear function for the unit price p of the t-shirt sold daily as a function of q (demand), the number t-shirts can be sold at unit price p.

Answer: When unit price $p = 20$, the demand $q = 50$ and when the unit price $p = 18.5$, the demand $q = 53$. The slope is $\frac{-1.5}{3} = -\frac{1}{2}$ dollar per t-shirt. Thus, $p - 20 = -\frac{1}{2}(q - 50)$, that is $p = -\frac{1}{2}q + 45$.

b. Calculate the consumers' surplus when the unit price is $\bar{p}$=15 dollars per shirt using the demand equation found in part (a). Consumers' Surplus is defined as $\int_0^{\bar{q}} (D(q) - \bar{p})dq$.

Answer: When $\bar{p} = 15$, then $15 = -\frac{1}{2}q + 45$, $-30 = -\frac{1}{2}q$, $\bar{q} = 60$.

$$CS = \int_0^{60} ((-\frac{1}{2}q + 45) - 15)dq = \int_0^{60} (-\frac{1}{2}q + 30)dq = (-\frac{1}{4}q^2 + 30q)\Big|_0^{60}$$

$$= -\frac{1}{4} \cdot 60^2 + 30 \cdot 60 = -900 + 1800 = \$\,900.$$

The consumers' surplus is 900 dollars for the first 60 t-shirt sold.

2. Calculate the producers' surplus for the supply equation at the indicated unit price $\bar{p} = \$\,160$ (Round your answer to the nearest cent.)

Producers" Surplus is defined a $\int_0^{\bar{q}} (\bar{p} - S(q))dq$ where $p = 130 + e^{0.01q}$, $\bar{p} = 160$.

Answer: When $\bar{p} = 160$, then $160 = 130 + e^{0.01q}$, $30 = e^{0.01q}$, $\bar{q} = \frac{\ln 30}{0.01} = 340.1197382$.

$$PS = \int_0^{\frac{\ln 30}{0.01}} (160 - (130 + e^{0.01q})) = \int_0^{\frac{\ln 30}{0.01}} (30 - e^{0.01q})dq = \left(30q - \frac{1}{0.01}e^{0.01q}\right)\Big|_0^{\frac{\ln 30}{0.01}}$$

$$= \left(30 \cdot \frac{\ln 30}{0.01} - \frac{1}{0.01} \cdot e^{0.01 \cdot \frac{\ln 30}{0.01}}\right) - \left(30 \cdot 0 - \frac{1}{0.01}e^{0.01 \cdot 0}\right) = \left(30 \cdot \frac{\ln 30}{0.01} - \frac{30}{0.01}\right) + \frac{1}{0.01} = 7303.59.$$

The producers' surplus is \$ 7303.59 for the first 340 items sold.

3. A company finds that the demand for their new product is given by $p = 13 - q^{\frac{1}{4}}$, where p is the price per item and q is the number of items that can be sold per week at unit price p. The company is prepared to sell $q = \left(\frac{p-4}{2}\right)^4$ items per week at a unit price p. Find the equilibrium price $\bar{p}$ and the consumers' and producers' surpluses at the equilibrium price. What is the total social gain at the equilibrium price?

Answer: To find the equilibrium price solve the following system of equations $p = 13 - q^{\frac{1}{4}}$ and $q = \left(\frac{p-4}{2}\right)^4$ simultaneously, that is find when the supply meets the demand. Substitute $q = \left(\frac{p-4}{2}\right)^4$ for q into $p = 13 - q^{\frac{1}{4}}$.

Using the identity $\left(a^k\right)^n = a^{k \cdot n}$, we obtain $p = 13 - q^{\frac{1}{4}} = 13 - \left(\frac{p-4}{2}\right)^{4 \cdot \frac{1}{4}} = 13 - \left(\frac{p-4}{2}\right)$. Solve the equation $p = 13 - \left(\frac{p-4}{2}\right)$ for p. Multiplying by 2, we get $2p = 26 - (p-4) = 30 - p$. Then $3p = 30$ and $\bar{p} = \$\,10$.

To find the corresponding demand $\bar{q}$, substitute $\bar{\boldsymbol{p}} = \mathbf{10}$ into $q = \left(\frac{p-4}{2}\right)^4$ and solve for q. Thus $\bar{\boldsymbol{q}} = \mathbf{81}$.

The equilibrium price is $\bar{\boldsymbol{p}} = \mathbf{10}$.

$$CS = \int_0^{81} \left(\left(13 - q^{\frac{1}{4}}\right) - 10\right) dq = \int_0^{81} \left(3 - q^{\frac{1}{4}}\right) dq = \left(3q - \frac{1}{1.25} q^{1.25}\right)\Big|_0^{81} = 3 \cdot 81 - \frac{1}{1.25} \cdot 81^{1.25} = \$\,48.6$$

The consumers' surplus for the first 81 items is $\mathbf{48.6}$ dollars.

Before we find the producers' surplus, we need to solve the supply equation for p. The 4th root of both sides of the equation $q = \left(\frac{p-4}{2}\right)^4$ to obtain $q^{\frac{1}{4}} = \left(\frac{p-4}{2}\right)$. Multiplying by 2 and adding 4 to both sides of the equation, we get $p = 2q^{1/4} + 4$. Let us calculate the producer surplus.

$$PS = \int_0^{81} \left(10 - \left(2q^{1/4} + 4\right)\right) = \int_0^{81} \left(6 - 2\,q^{1/4}\right) dq = \left(6q - \frac{2}{1.25} q^{1.25}\right)\Big|_0^{81} = 6 \cdot 81 - \frac{2}{1.25} 81^{1.25} = \$\,97.2$$

The producers' surplus for the first 81 items is $\mathbf{97.2}$ dollars.

The total social gain is the sum of CS and PS, that is $\mathbf{48.6 + 97.2 = 145.8}$ dollars for the first 81 items.

4. Given that x is number of items, the demand function is $d(x) = 200 - 0.2x$, and the supply function is $s(x) = 0.3x$.

   a) Find the equilibrium quantity.
   b) Find the consumers' surplus and producers' surplus at the equilibrium quantity.

   Answer: a) $x = 400$ items b) $CS = \$16{,}000$; $PS = \$24{,}000$

5. Given that x is number of items, the demand function is $d(x) = 270.4 - 0.1x^2$, and the supply function is $s(x) = 0.3x^2$.

   a) Find the equilibrium quantity.
   b) Find the consumers' surplus and producers' surplus at the equilibrium quantity.

   Answer: a) $x = 26$ items b) $CS = \$1{,}171.73$; $PS = \$3{,}515.2$

6. Given that x is number of items, the demand function is $d(x) = \frac{2304}{\sqrt{x}}$, and the supply function is $s(x) = 9\sqrt{x}$.

a) Find the equilibrium quantity.
b) Find the consumers' surplus and producers' surplus at the equilibrium quantity.

Answer: a) $x = 256$ items b) $CS = \$36{,}864$; $PS = \$12{,}288$

7. Your video store has the exponential demand of equation $p = 15e^{-0.01q}$, where q represents daily sales of used DVD's and p represents daily price you charge per DVD. Calculate the daily Consumer's Surplus if you sell used DVDs at \$5 each.

Answer: $CS = \int_0^{\bar{q}} (D(q) - \bar{p})dq = \int_0^{109.9} (15e^{-0.01q} - 5)dq = \$450.69$

8. Calculate the Producer's Surplus for the supply equation $p = 13 + 2q$ at the unit price $\bar{p} = 29$.

Answer: $PS = \int_0^{\bar{q}} (\bar{p} - S(q))dq = \int_0^{8} (29 - (13 + 2q))dq = \$64$

9. Calculate the Producer's Surplus for the supply equation $p = 7 + 2q^{\frac{1}{3}}$ at the unit price $\bar{p} = 14$.

Answer: $\bar{q} = 3.5^3 = 42.875$; $PS = \int_0^{\bar{q}} (\bar{p} - S(q))dq = \int_0^{42.875} \left(14 - (7 + 2q^{\frac{1}{3}})\right) dq = \$75$

Improper Integral (Section 14.5)

10. Determine whether the given improper integral converges or diverges.

- If convergent, evaluate the integral and give the numerical answer.
- If divergent, indicate if the integral diverges to positive infinity or negative infinity.

a) $\int_1^{\infty} \frac{8}{x^2} dx$

Answer: $\int_1^{\infty} \frac{8}{x^2} dx = \lim_{t\to\infty} \int_1^{t} \frac{8}{x^2} dx = \lim_{t\to\infty} \left(\left(\frac{-8}{x}\right)\Big|_1^t \right) = \lim_{t\to\infty} \left(-\frac{8}{t} + 8\right) = 8$ since $\lim_{t\to\infty} \left(-\frac{8}{t}\right) = 0.$

Thus, this improper integral converges to 8.

b) $\int_{-2}^{\infty} e^{-3x}\, dx$

Answer: $\int_{-2}^{\infty} e^{-3x}\, dx = \lim_{t\to\infty} \int_{-2}^{t} e^{-3x}\, dx = \lim_{t\to\infty} -\frac{1}{3}e^{-3x}|_{-2}^{t} = \lim_{t\to\infty} \left(-\frac{1}{3}e^{-3t} - \left(-\frac{1}{3}e^{6}\right)\right) =$

$\lim_{t\to\infty} \left(-\frac{1}{3}e^{-3t} + \frac{1}{3}e^{6}\right) = \frac{1}{3}e^{6} = \frac{e^6}{3}$ since the $\lim_{t\to\infty} \left(-\frac{1}{3}e^{-3t}\right) = 0.$

Thus, this improper integral converges to $\frac{e^6}{3}$.

c) $\int_1^{\infty} \frac{3}{x} dx$

Answer: $\int_1^{\infty} \frac{3}{x} dx = \lim_{t\to\infty} \int_1^{t} \frac{3}{x} dx = \lim_{t\to\infty} 3\ln|x|\,|_1^t = \lim_{t\to\infty} (3\ln|t| - 3\ln 1) = \infty$.

Since $\lim_{t\to\infty} \ln|t| = \infty$ and $\ln 1 = 0$.

Thus, this improper integral diverges.

d) $\int_1^{\infty} \frac{6}{x^2} dx$ — ***Answer*:** Converges to 6

e) $\int_{-\infty}^{-2} \frac{3}{x^2} dx$ — ***Answer*:** Converges to $\frac{3}{2}$

f) $\int_1^{\infty} x\, dx$ — ***Answer*:** Diverges to ∞

g) $\int_0^{3} \frac{1}{x^{1.1}} dx$ — ***Answer*:** Diverges to ∞

h) $\int_0^{3} \frac{1}{x^{0.1}} dx$ — ***Answer*:** Converges to 2.986528

i) $\int_{-\infty}^{-1} \frac{1}{x^{\frac{1}{3}}} dx$ — ***Answer*:** Diverges to - ∞

j) $\int_4^{\infty} \frac{1}{x^4} dx$ — ***Answer*:** Converges to 0.005208

k) $\int_0^{\infty} e^{-x}\, dx$ — ***Answer*:** Converges to 1

l) $\int_{-\infty}^{2} e^{x}\, dx$ — ***Answer*:** Converges to e^2

m) $\int_4^{\infty} e^{-2x}\, dx$ — ***Answer*:** Converges to $\frac{1}{2e^8}$

n) $\int_3^{\infty} x^2\, dx$ — ***Answer*:** Diverges to ∞

o) $\int_{-\infty}^{0} e^{2x}\, dx$ — ***Answer*:** Converges to $\frac{1}{2}$