MAT 275 TEST 1 PRACTICE

1. Two direction fields are plotted below

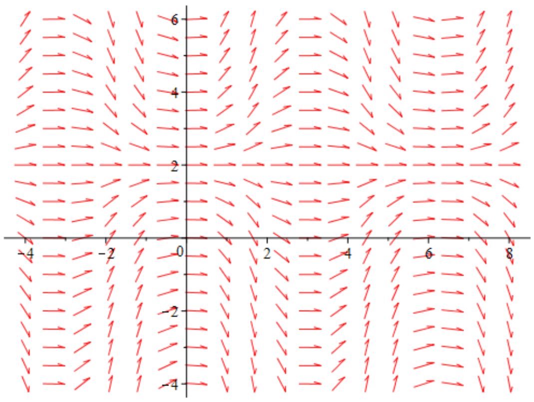


Figure 1

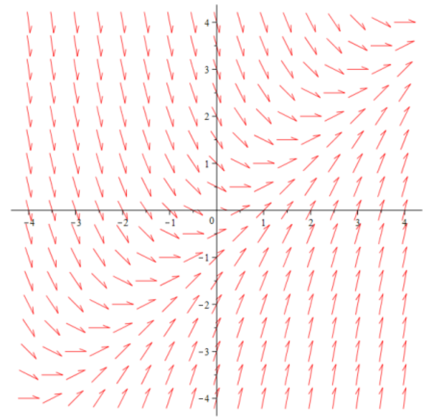


Figure 2

a) Which differential equation represents the slope field in **Figure 1**

(i) $y' = y\sin(x)$ (ii) $y' = x\sin(y)$ (iii) $y' = (y+2)\sin(x)$

(iv) $y' = (y-2)\sin(y)$ (v) $y' = (y-2)\sin(x)$

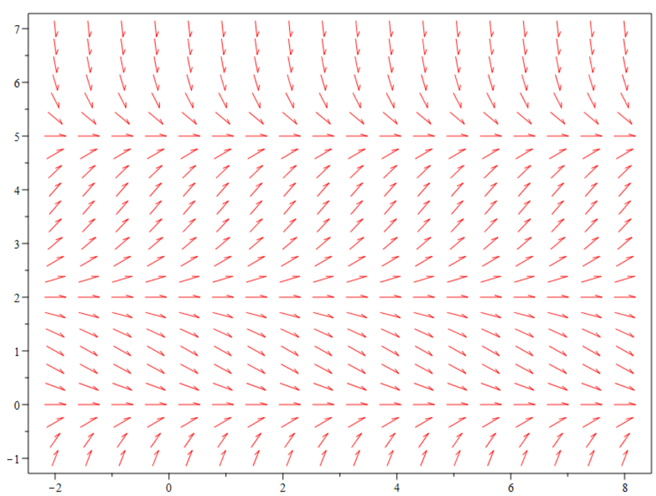
b) Which differential equation represents the slope field in **Figure 2**

(i) $y' = x + y$ (ii) $y' = x - y$ (iii) $y' = y - x$

(iv) $y' = xy$ (v) $y' = \frac{x}{y}$

c) In Figure 2, if $y(0) = 2$, estimate $y(1)$.

2. For the direction field plotted on the right:



a) Which of the following differential equations best represents the slope field at the right?

(i) $y' = 0.15y(y-5)(y-2)$

(ii) $y' = -0.15y(y-5)(y-2)$

(iii) $y' = 0.15(y-5)(y-2)$

(iv) $y' = -0.15y^2(y-5)(y-2)$

(v) $y' = -0.15y(y+5)(y+2)$

b) Label the equilibrium (constant) solutions as stable, unstable or semi-stable.

c) If $y(0) = 2.5$, estimate $y(2)$.

d) If $y(0) = 1$, what will be the behavior of the solution as $t \to \infty$?

e) If $y(0) = 2$, what will be the behavior of the solution as $t \to \infty$?

f) If $y(0) = 3$, what will be the behavior of the solution as $t \to \infty$?

3. For the ODE $y' = -y(y-3)(y-8)^2$,

a) determine all the equilibrium (constant) solutions and classify them as stable, unstable or semi-stable.

b) For what values of y is y increasing?

4. Consider the differential equation: $-4x + y^2 + 2xyy' = 0$.

(a) What is the order of the differential equation?

(b) Is the Differential Equation linear or nonlinear?

(c) Determine the value(s) of the constant A so that $y(x) = A\sqrt{x}$ is a solution to the Differential Equation.

5. Consider the differential equation: $x^2y'' - 3xy' + 5y = 2x^2\ln(x)$

(a) What is the order of the differential equation?

(b) Is the differential equation linear or nonlinear?

(c) Determine the value(s) of the constant A so that $y(x) = Ax^2\ln(x)$ is a solution to the differential equation.

6. A ball has mass 0.2 kg and is dropped from a tall tower and falls downward in the positive direction. We assume that the forces acting on the body are the force of gravity and a retarding force of air resistance with direction opposite to the direction of motion and with magnitude $cv(t)$ where c has units of $\frac{kg}{s}$ and $v(t)$ is the velocity of the ball in meters per second (mps) at time t. The gravitational constant is $g = 9.8\frac{m}{s^2}$. Find the value of c if the terminal velocity is 40 mps, and write the differential equation that describes the velocity at time t.

7. A stone has mass 5 kg and is dropped from a tall tower and falls downward in the positive direction. We assume that the forces acting on the body are the force of gravity and a retarding force of air resistance with direction opposite to the direction of motion and with magnitude $cv^2(t)$ where c has units of $\frac{kg}{m}$ and $v(t)$ is the velocity of the ball in meters per second (mps) at time t. The gravitational constant is $g = 9.8\frac{m}{s^2}$. Find the value of c if the terminal velocity is 50 mps, and write the differential equation that describes the velocity at time t.

8. Determine if each of the following equations is separable (Yes or No), and /or linear (Yes or No). Record your answer in the following table. Do not attempt to solve the equations.

Equation	Separable	Linear
$y' = \dfrac{t+1}{yt}$		
$y' = \dfrac{yt}{t+1}$		
$y' = \cos(ty)$		
$y' - ty = t^3$		

9. Solve the first order differential equation $y' = \frac{\cos(x)}{2y}, y(0) = -1$. Use your answer to find $y\left(\frac{\pi}{2}\right)$.

10. The general solution of the differential equation $xdy = ydx$ is a family of (determine the correct choice)

a) circles b) parabolas c) hyperbolas d) lines passing through the origin

11. Use separation of variables to find the solution of the following Initial Value Problems. Write your answer in explicit form and simplify as much as possible. For each IVP determine the interval in which the solution is defined.

a) $y' = \frac{2x}{y+1}, \quad y(1) = -2$

b) $xy^2 + 3y^2 - x^2y' = 0, \quad y(1) = 3$

c) $y' = \frac{2t}{t^2y+y}, \quad y(1) = 2$

d) $\frac{x^2}{y^2-3}\frac{dy}{dx} = \frac{1}{2y}, \quad y(1) = 2$

e) $\frac{dy}{dx} = 20yx^4, \quad y(0) = 4$

12. A completely filled 200 liter tank originally contains 20 kilograms of salt dissolved in water. Brine containing 0.3 kg of salt per liter enters the tank at the rate of 5 liters/minute, and the well-stirred mixture leaves the tank at the same rate. Find the amount of salt in the tank at any time *t*.

13. A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min; thus the tank is empty after exactly 1 hour. Let *y*(*t*) be the amount of salt in the tank after *t* minutes.

(a) Write an Initial Value Problem for the amount of salt in the tank at any time t (< 60).
(b) Solve the IVP in part (a) to find the amount of salt in the tank at any time t (< 60).
(c) Determine the amount of salt when the tank is half empty.

14. A ball with mass 0.2 kg is thrown upwards (in the positive direction) with initial velocity 26 meters per second. We assume that the forces acting on the body are the force of gravity and a retarding force of air resistance with direction opposite to the direction of motion and with magnitude $c|v(t)|$ where $c = 0.1\frac{kg}{s}$ and $v(t)$ is the velocity of the ball at time t. The gravitational constant is $g = 9.8\frac{m}{s^2}$.

(a) Write and solve the differential equation for the velocity, $v(t)$.
(b) Find the formula for the position function at any time t, if the initial release is taken to be $s(0) = 0$.
(c) Find the time it takes for the ball to reach its maximum height, and the time at which the ball returns to its initial height.

15. Newton's law of cooling is $u' = -k(u - T)$ where *u*(*t*) is the temperature of an object, *t* is in hours, *T* is a constant ambient temperature, and *k* is a positive constant. Suppose a building loses heat in accordance with Newton's law of cooling. Suppose that the rate constant *k* has the value $0.13hr^{-1}$. Assume that the interior temperature of the building is 76°F when the heating system fails and the external temperature is T=10°F.

(a) How long will it take for the interior temperature to fall to 32°F?
(b) What happens to the temperature *u*(*t*) as $t \to \infty$?

16. The population of a city increases continuously at a rate proportional, at any time, to the population at that time.

a) Set up a differential equation for the city.
b) Solve the differential equation if the initial population is 52,000, and the population doubles in 50yr.

c) Find the ratio of the population, P, to the initial population, P_0, after 75 years.

17) Suppose $P(t)$ denotes the size of an animal population at time t and its growth is described by the differential equation $\frac{dP}{dt} = 0.002P(1000 - P)$. Determine the value of P at which the population is growing fastest.

18) Solve the following Initial Value Problems using the method of integrating factor. Give the interval where each solution is guaranteed to have a unique solution.

a) $y' = \frac{2}{t}y + 6t^4\,, \qquad y(1) = -3$

b) $y' - 2y = 2e^{5t} + 5e^{2t}\,, \qquad y(0) = -3$

c) $ty' = -\frac{\sin(t)}{t} - 2y\,, \qquad y(\pi) = 1$

d) $(t+1)y' - 2y = 2t\,, \qquad y(0) = 4$

e) $\frac{dy}{dt} + 0.2ty = 5t\,, \qquad y(0) = 6$

19) Determine (without solving the problem) the maximal interval in which the solution of the given initial value problem is guaranteed to exist: $ty' + \tan(t)\,y = \sin(t)\,,\; y(\pi) = 6.$

20) a) Verify that both $y_1 = 2t - 1$ and $y_2 = t^2$ are solutions to $y' = 2\left(t - \sqrt{t^2 - y}\right)$.
In which intervals in t are the solutions valid?

b) Does the existence of two solutions of the given problem contradict any known theorem about existence and uniqueness of solutions to Differential Equations?

21) Consider the following differential equations. Determine if the Existence and Uniqueness Theorem does or does not guarantee existence and uniqueness of a solution of each of the following initial value problems.

I. $\frac{dy}{dx} = \sqrt{x - y},\; y(2) = 2$

II. $\frac{dy}{dx} = \sqrt{x - y},\; y(2) = 1$

III. $y\frac{dy}{dx} = x - 1,\; y(0) = 1$

IV. $y\frac{dy}{dx} = x - 1,\; y(1) = 0$

22. Use Euler's method with $n = 2$ steps for the differential equation $y' = te^y$, with initial value $y(1) = 0$, to find the approximate value of $y(1.5)$.

23. Use Euler's method with $n = 3$ steps for the differential equation $y' = 3t - 2y^2 - 2$, with initial value $y(1) = 1$, to find the approximate value of $y(1.6)$.

24. Which statement about Euler's method is false?

I. If you halve the step size, you approximately halve the error.

II. Euler's method never gives exact solutions.

III. Euler's method assumes that the slope of a solution curve is the same at all points in a short interval.

IV. Often, when applying Euler's method, the more steps you take the smaller the error.

V. Euler's method is used to string together a set of linearizations that approximate the curve.

25. Find a real valued solution to the following initial value problems. Sketch a graph of the solution.

a) $y'' + 3y' + 2y = 0$, with $y(0) = 3$, $y'(0) = 0$

b) $2y'' + 3y' - 2y = 0$, with $y(0) = 1$, $y'(0) = 7$

c) $y'' - 5y = 0$, with $y(0) = 1$, $y'(0) = -1$

d) $3y'' - 4y' = 0$, with $y(0) = 2$, $y'(0) = -8$

Section 3.2:

26. Determine the longest interval on which the given initial value problem is certain to have a unique twice differentiable solution.

(a) $(x-3)y'' + \frac{x}{x-3}y' + \sqrt{x-1}\,y = 0$, $\quad y(2) = 0$, $y'(2) = 1$

(b) $(t-1)y'' + ty' + y = \sec(t)$, $\quad y(0) = 1$, $y'(0) = 3$

(c) $t(t-4)y'' + 3y' + \ln(t)\,y = \sin(t)$, $\quad y(1) = 1$, $y'(1) = 1$

27. Which of the following pairs of functions is linearly independent on the entire real line?

A. $\{\sin(x),\ \cos(x)\}$ B. $\{e^x,\ xe^x\}$ C. $\left\{x,\ \left(\frac{e}{\pi}\right)^3 x\right\}$ D. $\{x,\ 3x\}$

E. $\{1,\ e^{-t}\}$ F. $\{\cos(t),\ \sin(t+\pi/2)\}$ G. $\{e^{-2t}\cos(2t),\ e^{-2t}\sin(2t)\}$

H. $\{2e^{-t},\ 4e^{-t+3}\}$ I. $\{e^{2t},\ e^{2t}-6\}$ J. $\{x,\ |x|\}$

28. Which of the following is NOT a fundamental set of solutions for $y'' - y = 0$?

A. $\{e^t,\ e^{-t}\}$ B. $\{2e^t,\ 2e^{-t}\}$ C. $\{te^t,\ e^{-t}\}$ D. $\left\{(e^t + e^{-t}),\ \frac{1}{2}(e^t + e^{-t})\right\}$

E. $\left\{\frac{1}{2}(e^t + e^{-t}),\ \frac{1}{2}(e^t - e^{-t})\right\}$ F. $\left\{\frac{1}{2}(e^t + e^{-t}),\ e^t\right\}$

29. Suppose $y_1(t) = t$ and $y_2(t) = t^2$ are both solutions of the second order linear equation $y'' + p(t)y' + q(t) = 0$. Which of the functions below are guaranteed to also be solutions of the same equation?

A. $y = t^2 - 1$ B. $y = 5t$ C. $y = -9t^2 + 17t$ D. $y = 0$

30. Consider the ODE $t^2y'' + 3ty' + y = 0$ with the initial conditions $y(1) = 1$, $y'(1) = 1$.

(i) What is the maximum interval of validity, I, of the solution?

(ii) Verify that the functions $y_1(t) = t^{-1}$ and $y_2(t) = t^{-1}\ln(t)$ satisfy the ODE for t in the interval I.

(iii) Find the Wronskian $W(y_1, y_2)$ to show that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions.

(iv) Solve the initial value problem.

Answers:

1. a) (v) b) (ii) c) $\cong 1.1$ to 1.2

2. a) (ii) b) y=0 stable, y=2 unstable, y=5 stable c) 4.5

d) $y \to 0$ e) $y \to 2$ f) $y \to 5$

3. (a) $y = 8$ semi-stable, $y = 3$ stable, $y = 0$ unstable (b) $0 < y < 3$

4. (a) first (b) non linear (c) $A = \pm\sqrt{2}$

5. (a) second order (b) linear (c) $A = 2$

6. $\frac{dv}{dt} = 9.8 - \frac{c}{0.2}v;\quad c = 0.049$

7. $\frac{dv}{dt} = 9.8 - \frac{c}{5}v^2;\quad c = 0.0196$

8.

Equation	Separable	Linear
$y' = \dfrac{t+1}{yt}$	Yes	No
$y' = \dfrac{yt}{t+1}$	Yes	Yes
$y' = \cos(ty)$	No	No
$y' - ty = t^3$	No	Yes

9. $y = -\sqrt{\sin(x) + 1};\ \ y\left(\frac{\pi}{2}\right) = -\sqrt{2}$

10. (d)

11.

a) $y = -1 - \sqrt{2x^2 - 1}$ Interval: $\left(\frac{1}{\sqrt{2}}, \infty\right)$

b) $y = \frac{-1}{\ln(x) - \frac{3}{x} + \frac{8}{3}} = \frac{-3x}{3x\ln(x) - 9 + 8x}$ Interval: $(0, 1.089)$

c) $y = -\sqrt{2\ln(t^2+1) + 4 - 2\ln(2)}$ Interval: $(-\infty, \infty)$

d) $y = \sqrt{3 + e^{1-\frac{1}{x}}}$ Interval: $(0, \infty)$

e) $y = 4e^{4x^5}$ Interval: $(-\infty, \infty)$

12. $y = 60 - 40e^{-t/40}$

13. (a) $\frac{dy}{dt} = 2 - \frac{3y}{60-t},\ \ y(0) = 0$

(b) $y(t) = 60 - t - \frac{(60-t)^3}{3600}$

(c) $y(30) = 22.5$ lbs

14. (a) $0.2\frac{dv}{dt} = -0.1v - (0.2)(9.8)$ or $\frac{dv}{dt} = -0.5v - 9.8;\quad v(t) = 45.6e^{-0.5t} - 19.6$

(b) $s(t) = 91.2 - 91.2e^{-0.5t} - 19.6t$

(c) $v = 0$ when $t = -2\ln\left(\frac{19.6}{45.6}\right) \cong 1.69$ sec. $s = 0$ when $t \cong 4.03$ sec.

15. (a) 8.45 hrs (b) $\lim_{t\to\infty} u(t) = 10$ ($u(t)$ will approach the external temperature).

16. (a) $\frac{dP}{dt} = rP$ (b) $P = 52000e^{0.013863t}$ (c) 2.83

17. $P = 500$

18.

(a) $y(t) = t^2(2t^3 - 5)$, for $t > 0$ (b) $y(t) = \left(\frac{2}{3}e^{3t} + 5t - \frac{11}{3}\right)e^{2t}$ for $-\infty < t < \infty$

(c) $y(t) = \frac{\cos(t)+\pi^2+1}{t^2}$, for $t > 0$ (d) $y(t) = -2t - 1 + 5(t+1)^2$, for $t > -1$

(e) $y(t) = 25 - 19e^{-t^2/10}$ for $-\infty < t < \infty$

Note: for (a) and (d), the solution looks good all reals, but uniqueness is lost if you go beyond the indicated interval.

19. $\frac{\pi}{2} < t < \frac{3\pi}{2}$

20. (a) $y_1(t)$ is a solution for $t \geq 1$; $y_2(t)$ is a solution for all t;
(b) f_y is not continuous at (1, 1).

21. Only II and III are guaranteed to have a unique solution.

22. $y(1.5) \approx 0.6513$

23. $y(1.6) \approx 1.005$

24. II (For example, for $y' = x - y$, the general solution is $y = -1 + x + Ce^{-x}$, so the solution to $y' = x - y,\ y(0) = -1$ is $y = -1 + x$, a line. So using Euler's method on $y' = x - y,\ y(0) = -1$ would give that same solution. See also **Figure 2** from #1.)

25. (a) $y(t) = -3e^{-2t} + 6e^{-t}$ (b) $y(t) = 4e^{t/2} - 3e^{-2t}$

(c) $y(t) = \frac{5-\sqrt{5}}{10}e^{\sqrt{5}t} + \frac{5+\sqrt{5}}{10}e^{-\sqrt{5}t}$ (d) $y(t) = 8 + 6e^{4t/3}$

26. (a) $1 < x < 3$ (b) $-\frac{\pi}{2} < t < 1$ (c) $0 < t < 4$

27. A, B, E, G, I, J

28. C: te^t is not a solution; D: the two functions are not linearly independent

29. By the principle of superposition, B, C, D

30. (i) $t > 0$

(iii) $W(y_1, y_2) = \frac{1}{t^3}$. Since the Wronskian is nonzero on I, $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions.

(iv) $y = \frac{1+2\ln(t)}{t}$