MAT 272 Test 2 Review

12.1 Planes and Surfaces

1. Find an equation of the plane containing the points (−6, −4, −4), (4,9, −2) and (7,5,3).
   a. 4x + 9y – 2z = −52
   b. −3x + 4y – 5z = 22
   c. −3x + 4y – 5z = 0
   d. 73x – 44y – 79z = 54
   e. None of the above

2. The equation $x^2 - y^2 - z^2 = 1$ represents
   a. A cone
   b. A hyperboloid of one sheet
   c. A hyperboloid of two sheets
   d. An ellipsoid
   e. None of the above.

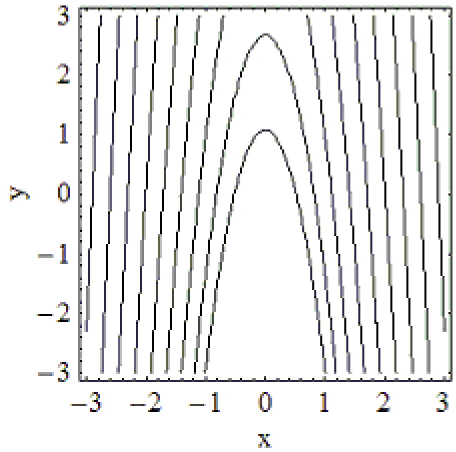
12.2 Graphs and Level Curves

1. For which of the following functions are the level curves corresponding to the positive integers evenly spaced circles?
   a. $f(x,y) = x^2 + y^2$
   b. $f(x,y) = \sqrt{x^2 + y^2}$
   c. $f(x,y) = x + y$
   d. $f(x,y) = (x^2 + y^2)^2$
   e. None of the above.

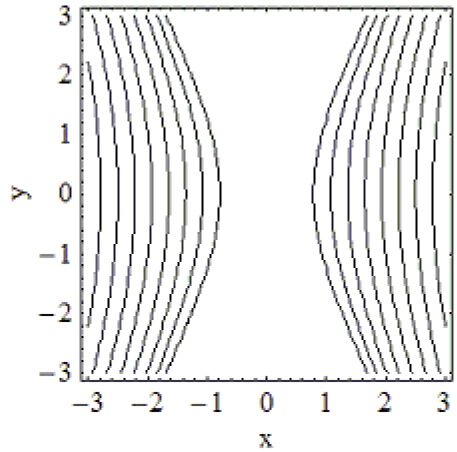
2. Identify the domain of the function $f(x,y) = \sqrt{x^2 + y^2 - 4} + \sqrt{9 - x^2 - y^2}$
   a. $\mathbb{R}^2$
   b. The closed annulus with outer radius 3 and inner radius 2.
   c. The open annulus with outer radius 3 and inner radius 2.
   d. The closed disk with radius $\frac{3}{2}$.
   e. The open disk with radius $\frac{3}{2}$.

3. Which of the following is a level curve plot of $(x, y) = \sqrt{4x^2 + y^2}$?

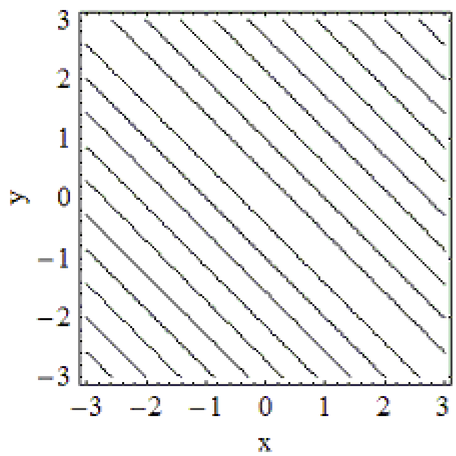
A.



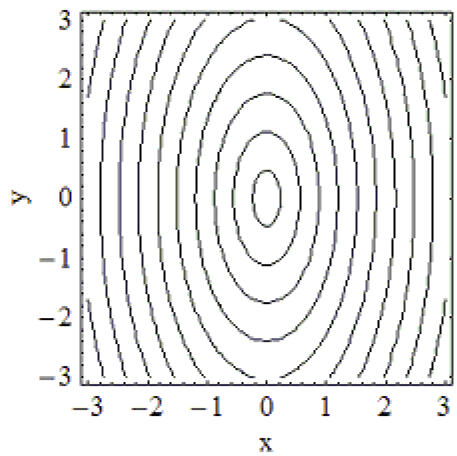
B.



C.



D.



12.3 Limits and Continuity

1. Does the function $f(x, y) = \frac{x+y}{x-y}$ have a limit as (x, y) goes to $(0,0)$?

   a. Yes, the limit is 1.

   b. Yes, the limit is -1.

   c. Yes, the limit is 0.

   d. Yes, the limit is a number other than 0, -1, 1.

   e. No, the limit doesn't exist.

2. Demonstrate that the following limit fails to exist:

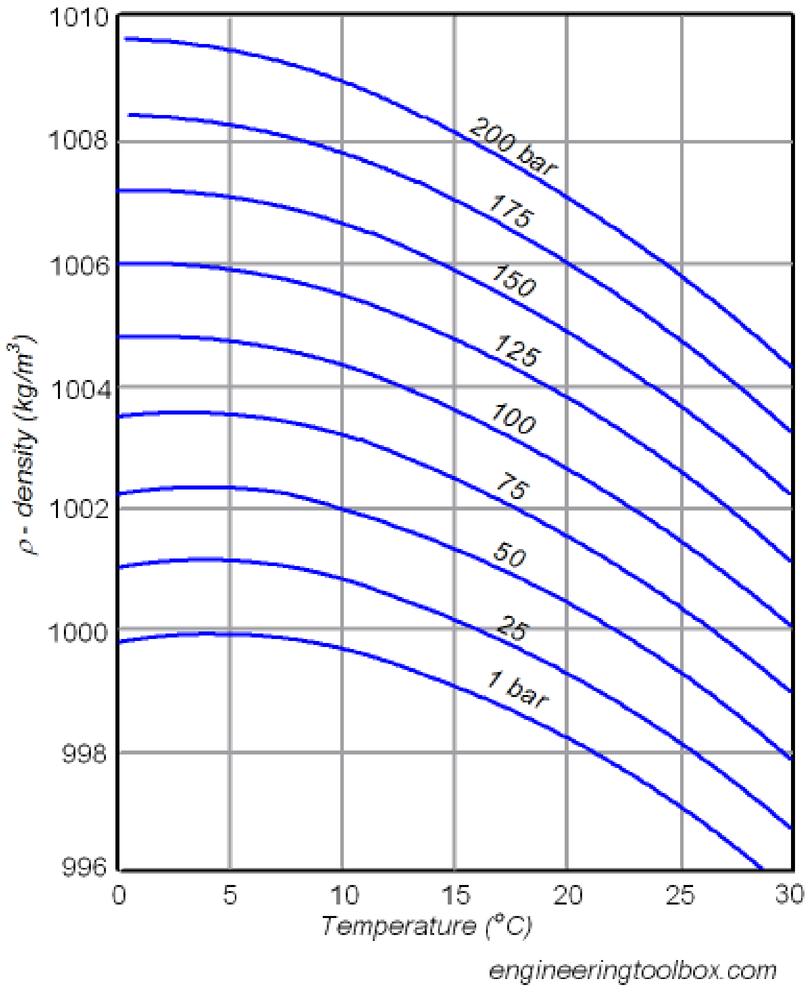
$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+2y^2}$$

12.4 Partial Derivatives

1. If f is a differentiable function, find the best estimates of $f_x(0,0)$ and $f_y(0,0)$ based on the following data: $f(0,0) = 1$, $f(0.1,0) = 2$ and $f(0,0.2) = 3$.
    a. $f_x(0,0) \approx 1$, $f_y(0,0) \approx 2$
    b. $f_x(0,0) \approx 2$, $f_y(0,0) \approx 3$
    c. $f_x(0,0) \approx 0.2$, $f_y(0,0) \approx 0.6$
    d. $f_x(0,0) \approx 10$, $f_y(0,0) \approx 10$

2. Suppose f is a differentiable function of two variables. What can you conclude based on the information $f_x(1,2) = 3$?
    a. The function f has a value of 3 at (1,2).
    b. From x=1 to x=2, the value of f changes by 3 units.
    c. A small displacement of the input point (1,2) by Δx in the x direction will result in an approximate change in function output by $3\Delta x$.
    d. At (1,2), if you go one unit step in the x direction, the function value will increase by <u>exactly</u> 3 units.
    e. None of the above.

3. A plot found on a website shows the water pressure (in bar) at which water has a given density (in kg per cubic meters) at a given temperature (in degrees Celsius). Thus, pressure P is a function of density ϱ and temperature T: $P = P(T, \varrho,)$.



In the following, let R be the point $(10°\,\mathrm{C}, 1002\ \mathrm{kg/m^3})$.

a. Estimate the partial derivative P_ϱ at R <u>using the 25 and 75 bar lines</u>.

b. Estimate the partial derivative P_T at R <u>using the 50 and 75 bar lines.</u>

12.5 The Chain Rule

1. Find $g'(t)$ where $g(t) = f(x(t), y(t))$, $f(x, y) = 2xy^2 + e^{2y}$, $x(t) = 2t^2 + 1$, $y(t) = t$.

2. Suppose h is a differentiable function of two variables x, y and f is a differentiable function of one variable t. Use the following tables to evaluate

(x,y)	$h(x,y)$	$h_x(x,y)$	$h_y(x,y)$
(1,0)	1	2	3
(1,1)	8	7	2
(2,0)	4	4	1
(2,1)	3	2	3

t	$f(t)$	$f'(t)$
1	2	-1
2	1	3
3	0	1
4	0	5

$g'(1)$, where $g(t) = h(f(t), t^2)$.

3. For a twice differentiable function $z = f(r, \theta)$ and polar coordinates r, θ, $r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}$,
   a. calculate the partial derivatives $\frac{\partial r}{\partial x}$ and $\frac{\partial \theta}{\partial x}$ and express them in terms of r and θ only.
   b. Express the partial derivative $\frac{\partial z}{\partial x}$ (with common abuse of notation) in terms of $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

12.6 Directional Derivatives and the Gradient

1. Find the direction of maximum change of $f(x, y) = e^{xy} + x^3 - 3y$ at the point (2,0).
   a. In the direction of $\frac{<6,-1>}{\sqrt{37}}$
   b. In the direction of $\frac{<12,-1>}{\sqrt{145}}$
   c. In the direction of $\frac{<3,-1>}{\sqrt{10}}$

d. In the direction of $\frac{<12,-3>}{\sqrt{154}}$

e. None of the above.

2. Which one of the following vectors $\vec{u}$ represents the direction of <u>steepest slope</u> of the function $f(x,y) = x^3 + y^4$ at the point (1,1), and what is this steepest slope m?

   a. $\vec{u} =< 3,4 >, m = 5,$

   b. $\vec{u} =< 3,4 >, m = 7$

   c. $\vec{u} =< 1,1 >, m = 2$

   d. $\vec{u} =< 1,1 >, m = 0$

   e. None of the above.

12.7 Tangent Planes and Linear Approximation

1. Find an equation of the tangent plane to the surface $z = xy - x - y$ at (-5,3,-13).

   a. $2(x + 5) - 6(y - 3) - z = 0$

   b. $2x - 6y - z = 0$

   c. $(x + 5) + (y - 3) - (z + 13) = 0$

   d. $2(x + 5) - 6(y - 3) - (z + 13) = 0$

   e. None of the above.

2. A company makes metal cylinders with a nominal radius of 5cm and a nominal height of 10cm. Use <u>differentials</u> to estimate the deviation of a cylinder's <u>volume</u> from the nominal value if the radius can be off by as much as 0.1cm, and the height can be off by as much as 0.2 cm.

   a. 0.3 cm^3

   b. 15.301π cm^3

   c. 48.073 cm^3

   d. 15π cm^3

   e. None of the above.

3. Find the best linear approximation of $f(4,2)$ if f is a differentiable function with $f(1,1) = 1$ and $f_x(1,1) = 5$ and $f_y(1,1) = 2$.
   a. $f(4,2) \approx 1$
   b. $f(4,2) \approx 8$
   c. $f(4,2) \approx 17$
   d. $f(4,2) \approx 18$
   e. None of the above.

12.8 Maximum/Minimum Problems

1. Given the function $f(x, y) = xy^3 + x^2y + 5x$, find all the critical points of f and apply the second derivative test to identify them as maximum, minimum or saddle points.
2. Find the maximum value of the function $f(x, y) = xy - y^2$ on the square $[0,1]^2$ (i.e. on the set where $0 \le x \le 1$ and $0 \le y \le 1$).

12.9 Lagrange Multipliers

1. Find the maximum of xy subject to the constraint $x^2 + y^2 \le 1$.
   a. 1
   b. 2
   c. ½
   d. The function $f(x, y) = xy$ does not have a maximum on the unit disk.
   e. None of the above.

2. Find the point on the line $x + y = 5$ that minimizes $f(x, y) = (x - 2)^2 + (y - 1)^2$.

3. What is the largest that the sum of the cubes of two real numbers can be if the sum of their squares is 1?

   a. 2
   b. 1
   c. $\frac{1}{\sqrt{2}}$

d. 0

Answers

12.1 Planes and Surfaces

1. 1 D
2. C

12.2 Graphs and Level Curves

1. B
2. B
3. D

12.3 Limits and Continuity

1. E
2. Let

$$f(x,y) = \frac{x^2+y^2}{x^2+2y^2}$$

Then $\lim\limits_{x\to 0} f(x,0) = 1$ but $\lim\limits_{y\to 0} f(0,y) = \frac{1}{2}$. Therefore $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+2y^2}$ fails to exist by the two-path test.

12.4 Partial Derivatives

1. D
2. C
3. We use difference quotients to approximate the partial derivatives.

   a. $P_Q \approx \frac{P(10,1003)-P(10,1001)}{1003-1001} = \frac{75-25}{2} = 25$

   b. $P_T \approx \frac{P(17.5,1002)-P(10,1002)}{17.5-10} = \frac{75-50}{7.5} = \frac{10}{3}$

12.5 The Chain Rule

1. We apply the chain rule

$$g'(t) = f_x\big(x(t), y(t)\big) \cdot x'(t) + f_y\big(x(t), y(t)\big) \cdot y'(t)$$

to find: $g'(t) = 2\big(y(t)\big)^2 \cdot 4t + \big(4x(t)y(t) + 2e^{2y(t)}\big) = 16t^3 + 4t + 2e^{2t}$

2. We apply the chain rule to $g(t) = h(f(t), t^2)$ and get

$$g'(t) = h_x(f(t), t^2) \cdot f'(t) + h_y(f(t), t^2) \cdot 2t$$

It follows that

$$g'(1) = h_x(f(1), 1) \cdot f'(1) + h_y(f(1), 1) \cdot 2$$

By using the data in the given tables we find

$$g'(1) = h_x(2,1) \cdot (-1) + h_y(2,1) \cdot 2 = 2 \cdot (-1) + 3 \cdot 2 = 4$$

3. For a twice differentiable function $z = f(r, \theta)$ and polar coordinates r, θ, $r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}$

   a.

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{r\cos\ \theta}{r} = \cos\ \theta$$

$$\frac{\partial \theta}{\partial x} = \left(\frac{1}{1 + \left(\frac{y}{x}\right)^2}\right) \cdot \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2 + y^2} = \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

   b. By applying the chain rule to the function $z = f(r(x, y), \theta(x, y))$ we find

$$\frac{\partial z}{\partial x} = f_r \frac{\partial r}{\partial x} + f_\theta \frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r}\cos\ \theta - \frac{\partial z}{\partial \theta}\frac{\sin\theta}{r}$$

12.6 Directional Derivatives and the Gradient

1. B.
2. A

12.7 Tangent Planes and Linear Approximation

1. D
2. D
3. D

12.6 Maximum/Minimum Problems

1. Critical points (x,y) solve $f_x = y^3 + 2xy + 5 = 0$ and $f_y = 3xy^2 + x^2 = 0$. The second equation is equivalent to $x(3y^2 + x) = 0$ which means that $x = 0$ or $3y^2 + x = 0$.
   If $x = 0$ then the condition $y^3 + 2xy + 5 = 0$ simplifies to $y^3 + 5 = 0$ which means that $y = \sqrt[3]{-5}$.
   If $3y^2 + x = 0$ then $x = -3y^2$. Substituting this into $y^3 + 2xy + 5 = 0$ yields $y^3 - 6y^3 + 5 = 0$ which is solved by $y = 1$.

   Thus the critical points are $(0, \sqrt[3]{-5})$ and $(-3,1)$. We now apply the second derivative test. With

   $f_{xx} = 2y$, $f_{xy} = 3y^2 + 2x$, $f_{yy} = 6xy$

   we compute the discriminant to be

   $$D = 2y \cdot 6xy - (3y^2 + 2x)^2 = 12xy^2 - (3y^2 + 2x)^2$$

   .

   By plugging in the two critical points, we recognize that D is negative at both of them. Therefore, both critical points are saddle points.

2. We need to collect three types of candidate points for the location of the absolute max: critical points of the surface, critical points of its boundary curves and corner points.

a. The system $f_x = y = 0$, $f_y = x - 2y = 0$ has only one solution: (0,0). This is the sole critical point of the surface, and it is in the domain.

b. Collecting the critical points of the boundary curves.
   i. The boundary curve $z = f(0, y) = -y^2$ has derivative $\frac{dz}{dy} = -2y$ and therefore a critical point at $y = 0$.
   ii. The boundary curve $z = f(1, y) = y - y^2$ has derivative $\frac{dz}{dy} = 1 - 2y$ and therefore a critical point at $y = \frac{1}{2}$.
   iii. The boundary curve $z = f(x, 0) = 0$ is constant. Every one of its points is critical.
   iv. The boundary curve $z = f(x, 1) = x$ is linear and non-constant and therefore contains no critical points.

c. The corner points are (0,0), (0,1), (1,0) and (1,1).

We now make a table of function values for the points discovered.

(x,y)	f(x,y)
(x,0)	0
(0,1)	-1
(1,1)	0
(1,0.5)	.25

Answer: the maximum value is $\frac{1}{4}$, achieved at the point $(1, \frac{1}{2})$.

12.9 Lagrange Multipliers

1. C

2. Find the point on the line $x + y = 5$ that minimizes $f(x, y) = (x - 2)^2 + (y - 1)^2$.

We have to solve the system of equations $f_x = \lambda g_x$ and $f_y = \lambda g_y$ where $g(x, y) = x + y$. By substituting the partial derivatives, we get $2(x - 2) = \lambda$ and $2(y - 1) = \lambda$. The Lagrange multiplier is very conveniently eliminated here: $2(x - 2) = 2(y - 1)$ or $x - 1 = y$. We substitute $y = 5 - x$ into that equation to eliminate y: $x - 1 = 5 - x$. This leads to the solution $(x, y, \lambda) = (3,2,2)$. Since the question guarantees the existence of a minimum of f on the line $x + y = 5$, it must be at (3,2).

3. B

