MAT 272 Test 1 Review

11.1 Vectors in the Plane

1. Let $P = (1,1)$ and $Q = (2,3)$. Find the unit vector $\vec{u}$ that has the same direction as $\overrightarrow{QP}$.
   a. $\vec{u} =< -1, -2 >$
   b. $\vec{u} =< \frac{-1}{\sqrt{5}}, -\frac{-2}{\sqrt{5}} >$
   c. $\vec{u} =< 1, 2 >$
   d. $\vec{u} =< \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} >$
   e. None of the above.

2. If $\vec{u}$ has magnitude 3, $\vec{v}$ has magnitude 4, and the direction of $\vec{v}$ is perpendicular to the direction of $\vec{u}$, what is the magnitude of $\vec{u} + \vec{v}$?

3. A boat is at A and needs to get to B, which is 40 miles due north of A. The wind is moving 10 miles per hour on a bearing of 10 degrees N of E, and the current is moving to the southeast (45 degrees S of E) at 15 miles per hour. Assume the boat needs to make the journey from A to B in exactly one hour. State the (a) speed and (b) the bearing (angle in degrees and a direction) at which the boat must travel so that combined with the wind and current, will lead directly to B.

11.2 Vectors in Three Dimensions

1. Find an equation of the sphere with radius 4, and center (–8, 2, 5) .
   a. $(x + 8)^2 + (y - 2)^2 + (z - 5)^2 = 4$
   b. $(x + 8)^2 + (y - 2)^2 + (z - 5)^2 = 16$
   c. $(x - 8)^2 + (y + 2)^2 + (z + 5)^2 = 16$
   d. $(x - 8)^2 + (y + 2)^2 + (z + 5)^2 = 4$
   e. None of the above.

2. Identify radius and center of the sphere $x^2 + 2x + y^2 - 4y + z^2 - 10z = 91$.

11.3 Dot Products

1. The exact angle between the vectors $< 1,2,2 >$ and $< 2,-1,2 >$ is..
   a. $\arccos\frac{4}{9}$
   b. $\frac{<2,-2,4>}{9}$
   c. $\arcsin\frac{4}{9}$
   d. $\frac{4}{9}$
   e. None of the above

2. The dot product of a magnitude 4 vector and a magnitude 7 vector is..
   a. always 28.
   b. always in the interval [0, 28].
   c. always in the interval [-28,28].
   d. always in the interval [4,7].
   e. None of the above.

3. Suppose a ramp has a slope of 10%. A crate sitting on the ramp has a weight of 100N. Use projection to find the magnitude of the force pressing the crate against the ramp. Do **NOT** use trigonometry.

11.4 Cross Products

1. Find a unit vector orthogonal to $< -2,2,5 >$ and $< 2,8,4 >$.

2. The area of the triangle spanned by the vectors $\vec{u} =< 4,2,0 >$ and $\vec{v} =< 0,1,3 >$ is ..
   a. 4
   b. 6
   c. 7
   d. 14
   e. None of the above.

3. If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^3$, and $\vec{u}\times\vec{v}=\vec{u}\times\vec{w}$, then ..
   a. $\vec{v}=\vec{w}$
   b. $\vec{v}=-\vec{w}$
   c. $\vec{v}$ and $\vec{w}$ must differ by a vector that is parallel or anti-parallel to $\vec{u}$.
   d. $\vec{v}$ and $\vec{w}$ must differ by a vector that is orthogonal to $\vec{u}$.
   e. None of the above.

11.5 Lines and Curves in Space

1. The straight line through the points (0,1) and (4,3) is described by..
   a. $\vec{r}(t) =< 4,3 > +t < -4,-2 >$
   b. $\vec{r}(t) =< 4,3 > +t < 4,2 >$
   c. $\vec{r}(t) =< 0,1 > +t < 4,2 >$
   d. $\vec{r}(t) =< 2,2 > +t < 2,1 >$
   e. All of the above describe the indicated straight line.

2. Evaluate

$$\lim_{t\to 0} < e^t, t^2-9, \frac{t+3}{t-9} >$$

   or explain why it does not exist.

11.6 Calculus of Vector-Valued Functions

1. Find the derivative of the vector-valued function $\vec{r}(t) =< \sqrt{t-4}, t^4, e^{-6t} >$.
2. Find the indefinite integral of $< \frac{1}{t}+9, 3t+8 >$.

11.7 Motion in Space

1. Find the position function for the given acceleration function and initial conditions:
   $\vec{a}(t) =< 4\sin(2t), 8\cos(2t), e^{-t} >, \vec{v}(0) =< 0,0,0 >, \vec{r}(0) =< 0,0,0 >$.

2. A rock is thrown from the top of a building 150 m high, with an initial speed of 40 meters per second at an angle of 55 degrees from the horizontal. Assume gravitational acceleration is –9.8 m/s^2. Assume no air resistance or other forces except for gravity. How far away (horizontally) does the rock land, and what is its impact speed when it hit the ground?

3. Suppose a particle moves on a sphere centered at the origin in $\mathbb{R}^3$. This means that if $\vec{r}(t)$ is the vector function that describes the particle's trajectory, then $|\vec{r}(t)|$ is constant. What can you say about the velocity of the particle?
   a. The particle must move at constant velocity.
   b. The particle must move at constant speed.
   c. The particle's velocity must be orthogonal to its position vector at all times.
   d. The particle's trajectory is a circle.
   e. None of the above.

11.8 Length of Curves

1. Find the exact length of the curve $\vec{r}(t) =< 3\cos t\,, 3\sin t\,, \frac{2}{3}t^{\frac{3}{2}} >, 0 \le t \le 2$.
2. Evaluate the arc length of the curve $\vec{r}(t) =< t, \frac{4}{3}\sqrt{t^3}, t^2 >, 0 \le t \le 1$.

11.9 Curvature and Normal Vectors

1. For the helix $\vec{r}(t) =< \cos 3t\,, \sin 3t\,, 4t >$, find the unit tangent vector $\vec{T}$, the unit normal vector $\vec{N}$, the curvature κ and the normal acceleration component a_N at t=0.

2. A particle's position vector traces out a parabola and is given by $\vec{r}(t) =< t, t^2 >$. Determine the curvature of this path at a general time t, and the radius of the circle of osculation at the origin.

3. If the <u>tangential</u> acceleration component of a <u>moving</u> object is constant zero, then..
   a. The object must be moving on a straight line at constant speed.
   b. The object must be moving on a straight line, but not necessarily at constant speed.
   c. The object must be moving on a circle.
   d. The object may not be moving at constant velocity, but it is moving at constant speed.
   e. None of the above has to be true.

Answers

11.1 Vectors in the Plane

1. B.
2. $|\vec{u}+\vec{v}| = \sqrt{3^2+4^2} = 5$.
3. The wind velocity is $\vec{w} = 10 < \cos 10^\circ, \sin 10^\circ >$, the current velocity is $\vec{c} = 15 < -\cos 45^\circ, -\sin 45^\circ >$. If the boat's own velocity is $\vec{v}$, then we need $\vec{w}+\vec{c}+\vec{v} =< 0,40 >$. Solving for $\vec{v}$ we get $\vec{v} =< 0.758,48.87 >$. By computing magnitude and direction of that vector, we find that the boat must travel at a speed of 48.876 miles per hour on a bearing of .889 degrees E of N.

11.2 Vectors in Three Dimensions

1. B.
2. We complete the squares to rewrite the equation as
$$(x+1)^2+(y-2)^2+(z-5)^2 = 121$$
   The radius is therefore 11, and the center is (-1,2,5).

11.3 Dot Products

1. A
2. C.

3. The gravitational force acting on the crate is $\vec{F} =< 0, -100 >$ N. Using the slope of the ramp of 10% and similar triangles, we determine that the vector $\vec{v} =< 1, -10 >$ represents the downward orthogonal direction to the ramp's incline. The magnitude F of the force pressing the crate against the ramp is the scalar projection of $\vec{F}$ onto $\vec{v}$:

$$F = \text{scal}_{\vec{v}:}\vec{F} = \frac{\vec{F} \cdot \vec{v}}{|\vec{v}|^2} = \frac{1000N}{\sqrt{101}} \approx 99.504\ N$$

11.4 Cross Products

1. By taking plus or minus the cross product of the two given vectors and dividing the resulting vector by its magnitude, we find that the two possible solutions are $< -\frac{16}{\sqrt{437}}, \frac{9}{\sqrt{437}}, -\frac{10}{\sqrt{437}} >$ and $< \frac{16}{\sqrt{437}}, -\frac{9}{\sqrt{437}}, \frac{10}{\sqrt{437}} >$.
2. C
3. C

11.5 Lines and Curves in Space

1. E.
2. $\lim_{t\to 0} < e^t, t^2 - 9, \frac{t+3}{t-9} >=< 1, -9, 3 >$. All component functions are continuous at $t = 0$. By definition of continuity, the limit is therefore equal to the value of the vector function at $t = 0$.

11.6 Calculus of Vector-Valued Functions

1. We find the derivative of the vector function by differentiating each component function separately:

$$\vec{r}'(t) =< \frac{1}{2\sqrt{t-4},} 4t^3, -6e^{-6t} >$$

2. We find the indefinite integral of the vector function by integrating each component function separately. Each component indefinite integral contains a free additive constant which leads to a free additive vector constant:

$$\int < \frac{1}{t} + 9, 3t + 8 > dt = < \ln|t| + 9t, \frac{3}{2}t^2 + 8t > + \vec{C}$$

11.7 Motion in Space

1. By integrating $\vec{a}(t)$ we find $\vec{v}(t) =< -2\cos(2t), 4\sin(2t), -e^{-t} > +\vec{C}$. The initial condition $\vec{v}(0) =< 0,0,0 >$ is satisfied for $\vec{C} =< 2,0,1 >$. We integrate again:
$$\vec{v}(t) =< -2\cos(2t) + 2, 4\sin(2t), -e^{-t} + 1 >$$
$$\vec{r}(t) =< -\sin(2t) + 2t, -2\cos(2t), e^{-t} + t > +\vec{D}$$
The initial condition $\vec{r}(0) =< 0,0,0 >$ is satisfied for $\vec{D} =< 0,2,-1 >$. Therefore,
$$\vec{r}(t) =< -\sin(2t) + 2t, -2\cos(2t) + 2, e^{-t} + t - 1 >.$$

2. This is an initial value problem with $\vec{a}(t) =< 0, -9.8 >$, $\vec{v}(0) = 40 < \cos 55^\circ, \sin 55^\circ >$and $\vec{r}(0) =< 0,150 >$ using a coordinate system with origin at the bottom of the building, the x-axis representing the horizontal and the y-axis representing the vertical.

   Integrating twice and taking the initial condition into account each time, we find
$$\vec{v}(t) =< 40\cos 55^\circ, -9.8t + 40\sin 55^\circ >$$
   and
$$\vec{r}(t) =< 40t\cos 55^\circ, -4.9t^2 + 40t\sin 55^\circ + 150 >$$

   By setting the y component to zero and solving for t, we find the ground impact time to be $t = 9.808$ seconds. By evaluating the first component at that time, we determine the horizontal distance to be 225.027 meters. The impact velocity is $\vec{v}(9.808) =< 22.943, -63.353 >$ and the impact speed is 67.38 m/s.

3. C

11.8 Length of Curves

In both problems, we use the arc length formula $s = \int_a^b |\boldsymbol{r}'(t)dt|$.

1. $s = \int_0^2 \sqrt{9+t}\,dt = \frac{2}{3}(5^{\frac{3}{2}} - 3^{\frac{3}{2}})$

2. $s = \int_0^1 \sqrt{1+4t+4t^2}\,dt = \int_0^1 (2t+1)dt = 2$

11.9 Curvature and Normal Vectors

1. By differentiating, we find $\vec{r}'(t) =< -3\sin 3t\,, 3\cos 3t\,, 4 >$. Therefore $\vec{T}(t) ==< -\frac{3}{5}\sin 3t\,, \frac{3}{5}\cos 3t\,, \frac{4}{5} >$ and thus $\vec{T}(0) =< 0, \frac{3}{5}, \frac{4}{5} >$.

   To find $\vec{N}$, we differentiate $\vec{T}(t)$:
   $$\vec{T}'(t) =< -\frac{9}{5}\cos 3t\,, -\frac{9}{5}\sin 3t\,, 0 >$$
   $\vec{N}$ is the unit vector version of that, so $\vec{N}(0) =< -1,0,0 >$.

   To find a_N and κ we calculate $\vec{r}''(t) =< -9\cos 3t\,, -9\sin 3t\,, 0 >$ and use the cross product formula:
   $$a_N(0) = \frac{|\vec{r}''(0) \times \vec{r}'(0)|}{|\vec{r}'(0)|} = \frac{| < -9,0,0 > \times < 0,3,4 > |}{| < 0,3,4 > |} = \frac{45}{5} = 9$$

   Since $\kappa = \frac{a_N}{v^2}$,
   $$\kappa = \frac{9}{25}$$

2. In order to be able to apply the cross product formula for curvature, we think of the particle's two dimensional movement as happening in the xy plane in $\mathbb{R}^3$. Thus, $\vec{r}(t) =< t, t^2, 0 >$. Then $\vec{r}'(t) =< 1,2t,0 >$ and $\vec{r}''(t) =< 0,2,0 >$. It follows that
   $$\kappa(t) = \frac{|< 0,2,0 > \times < 1,2t,0 >|}{|< 1,2t,0 >|^3} = \frac{2}{(1+4t^2)^{\frac{3}{2}}}$$
   The radius $R(0)$ of the circle of osculation at the origin is the reciprocal of $\kappa(0) = 2$: $R(0) = \frac{1}{2}$.

3. D

