Alternating Series

A series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1} a_n + \cdots$$

converges if

$$0 < a_{n+1} < a_n \quad \text{for all } n \quad \text{and} \quad \lim_{n\to\infty} a_n = 0.$$

Example. Show that the following alternating harmonic series converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

Series of Both Positive and Negative Terms

Theorem: Convergence of Absolute Values Implies Convergence

If $\sum |a_n|$ converges, then so does $\sum a_n$.

Explain how we know that the following series converges

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \cdots.$$

We say that the series $\sum a_n$ is

- **absolutely convergent** if $\sum a_n$ and $\sum |a_n|$ both converge.
- **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Test for convergence.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n-4}{\sqrt{n^3+n^2+8}}$
3. $\displaystyle\sum_{n=2}^{\infty} \frac{n\ln n + 4}{n^2}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n \sin^2 n}{n^3+1}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$

8. $\sum \frac{(-1)^n}{n^4+7}$

9. $\sum \frac{(-1)^{n-1}}{n \ln n}$

10. $\sum_{n=1}^{\infty} \frac{1}{n^2} \tan\left(\frac{1}{n}\right)$

11. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n}{n^2}$

12. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{2^n}$

Power Series

A **power series** about $x = a$ is a sum of constants times powers of $(x - a)$:

$$C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots + C_n(x-a)^n + \cdots = \sum_{n=0}^{\infty} C_n(x-a)^n.$$

A power series may converge for some values of x and not for others.

Intervals of Convergence

Each power series falls into one of the three following cases, characterized by its *radius of convergence*, R.

- The series converges only for $x = a$; the **radius of convergence** is defined to be $R = 0$.
- The series converges for all values of x; the **radius of convergence** is defined to be $R = \infty$.
- There is a positive number R, called the **radius of convergence**, such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$. See Figure 9.11.
- The **interval of convergence** is the interval between $a - R$ and $a + R$, including any endpoint where the series converges.

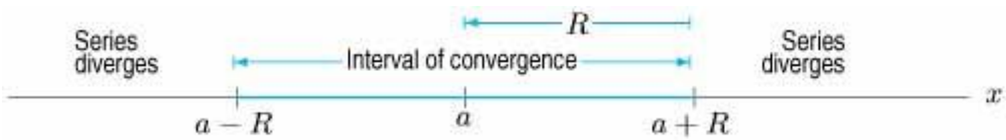

Figure: Radius of convergence, R, determines an interval, centered at $x = a$, in which the series converges

Theorem: Method for Computing Radius of Convergence

To calculate the radius of convergence, R, for the power series $\sum_{n=0}^{\infty} C_n(x-a)^n$, use the ratio test with $a_n = C_n(x - a)^n$.

- If $\lim_{n\to\infty} |a_{n+1}|/|a_n|$ is infinite, then $R = 0$.
- If $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 0$, then $R = \infty$.
- If $\lim_{n\to\infty} |a_{n+1}|/|a_n| = K|x-a|$, where K is finite and nonzero, then $R = 1/K$.

Determine radius of convergence and the interval o convergence of the following power series:

1. $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$
2. $\sum_{n=2}^{\infty} \frac{(-1)^n(x-2)^{2n}}{n^2}$.
3. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$.
4. $(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)}{4}^{4} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots$.
5. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$
6. $1 + 2^2x^2 + 2^4x^4 + 2^6x^6 + \cdots + 2^{2n}x^{2n} + \cdots$.
7. $2(x+5)^3 + 3(x+5)^5 + \frac{4(x+5)^7}{2!} + \frac{5(x+5)^9}{3!} + \cdot\cdot\cdot$
8. $\sum_{n=1}^{\infty} \frac{2^n(x-1)^n}{n}$