MAT 242 Test 2 SOLUTIONS, FORM A

1. [30 points] For the matrix A below, find a basis for the null space of A, a basis for the row space of A, a basis for the column space of A, the rank of A, and the nullity of A. The reduced row echelon form of A is the matrix R given below.

$$A = \begin{bmatrix} 5 & 15 & 5 & 0 & 4 \\ 4 & 12 & 4 & 5 & -3 \\ -2 & -6 & -2 & 0 & -2 \\ -2 & -6 & -2 & 1 & -5 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: To find a basis for the null space, you need to solve the system of linear equations $A\vec{x} = \vec{0}$, or equivalently $R\vec{x} = \vec{0}$. Parameterizing the solutions to this equation produces

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3\alpha - \beta \\ \alpha \\ \beta \\ 0 \\ 0 \end{bmatrix} = \alpha \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

so $\boxed{\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}}$ *is a basis for the* **null space**. *The* **nullity** *is the number of vectors in this basis, which is* $\boxed{2.}$

A basis for the **row space** *can be found by taking the nonzero rows of R:*

$$\boxed{\{[1,\ 3,\ 1,\ 0,\ 0], [0,\ 0,\ 0,\ 1,\ 0], [0,\ 0,\ 0,\ 0,\ 1]\}}$$

A basis for the **column space** *can be found by taking the columns of A which have pivots in them, so* $\boxed{\left\{ \begin{bmatrix} 5 \\ 4 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ -2 \\ -5 \end{bmatrix} \right\}}$ *is a basis for the column space.*

The **rank** *of A is the number of vectors in a basis for the row space (or column space) of A, so the rank of A is* $\boxed{3.}$

Grading: +10 points for finding a basis for the null space, +5 points for each of: a basis for the row space, a basis for the column space, the nullity, the rank. Grading for common mistakes: −3 points for forgetting a variable in the parameterization; −3 points for choosing columns of R for the column space of A; −3 points for choosing rows from A for the row space of A; −7 points for choosing the non-pivot columns of A for the null space of A.

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2. Let B be the (ordered) basis $\left(\begin{bmatrix}-1\\-1\\-3\end{bmatrix},\begin{bmatrix}1\\2\\5\end{bmatrix},\begin{bmatrix}-2\\-3\\-7\end{bmatrix}\right)$ and C the basis $\left(\begin{bmatrix}-1\\-3\\-2\end{bmatrix},\begin{bmatrix}-3\\-8\\-3\end{bmatrix},\begin{bmatrix}1\\3\\3\end{bmatrix}\right)$.

a. [10 points] Find the coordinates of $\begin{bmatrix}-2\\0\\0\end{bmatrix}$ with respect to the basis B.

Solution:

$$[\vec{u}]_B = \widetilde{B}^{-1}\cdot\vec{u} = \begin{bmatrix}-1 & 1 & -2\\-1 & 2 & -3\\-3 & 5 & -7\end{bmatrix}^{-1}\cdot\begin{bmatrix}-2\\0\\0\end{bmatrix} = \boxed{\begin{bmatrix}2\\4\\2\end{bmatrix}}.$$

Grading: +3 points for the formula, +4 points for substitution, +3 points for calculation.

b. [10 points] If the coordinates of $\vec{u}$ with respect to B are $\begin{bmatrix}1\\2\\3\end{bmatrix}$, what are the coordinates of $\vec{u}$ with respect to C?

Solution:

$$[\vec{u}]_C = \widetilde{C}^{-1}\cdot\widetilde{B}\cdot[\vec{u}]_B = \begin{bmatrix}-1 & -3 & 1\\-3 & -8 & 3\\-2 & -3 & 3\end{bmatrix}^{-1}\cdot\begin{bmatrix}-1 & 1 & -2\\-1 & 2 & -3\\-3 & 5 & -7\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} = \boxed{\begin{bmatrix}-53\\9\\-31\end{bmatrix}}.$$

Grading: +3 points for the formula, +4 points for substitution, +3 points for calculation. Grading for common mistakes: +7 points for $\begin{bmatrix}-27\\19\\25\end{bmatrix}$ *(backwards), +5 points for* $CB^{-1}[u] = \begin{bmatrix}-1\\-4\\-7\end{bmatrix}$, *+5 points for* $BC^{-1}[u] = \begin{bmatrix}-15\\-20\\-51\end{bmatrix}$, *+5 points for* $C^{-1}[u] = \begin{bmatrix}6\\-1\\4\end{bmatrix}$, *+7 points for finding the change-of-basis matrix.*

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3. [15 points] Let $\vec{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ -3 \\ 5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 24 \\ 3 \\ -6 \\ 21 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix}$. Is the vector $\begin{bmatrix} 0 \\ -2 \\ 2 \\ -5 \end{bmatrix}$ in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$? Justify your answer.

Solution: You must determine whether the augmented matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4|\vec{u}]$ *is a system that has at least one solution.*

$$\left[\begin{array}{rrrr|r} 5 & 3 & 24 & 2 & 0 \\ 0 & 1 & 3 & 1 & -2 \\ 1 & -3 & -6 & 2 & 2 \\ 2 & 5 & 21 & 2 & -5 \end{array}\right] \xrightarrow[RREF]{} \left[\begin{array}{rrrr|r} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since this system has infinitely many solutions, it has at least one, and so the vector $\boxed{\text{is in the span.}}$

Grading: +4 points for setting up the matrix, +4 points for the RREF, +3 points for determining how many solutions there were, +4 points for answering YES/NO. Grading for common mistakes: −8 points for using $\vec{0}$ *instead of* $\vec{u}$.

4. [15 points] Find a basis for the subspace spanned by $\left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -18 \\ 20 \\ -4 \\ 16 \end{bmatrix}, \begin{bmatrix} 15 \\ -11 \\ -8 \\ -7 \end{bmatrix} \right\}$, and the dimension of that subspace.

Solution: To find this basis, use the column space (CS) approach: Glue the vectors together to get the matrix $\widetilde{V}$, *find the RREF, and take the original vectors that have pivots in their columns:*

$$\widetilde{V} = \begin{bmatrix} 1 & -4 & -2 & -18 & 15 \\ 3 & 2 & 3 & 20 & -11 \\ -4 & 3 & -4 & -4 & -8 \\ 3 & 1 & 3 & 16 & -7 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 4 & -4 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\boxed{\left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -4 \\ 3 \end{bmatrix} \right\}}$ *is a basis for this subspace. Its dimension is the number of vectors in the basis, which is* $\boxed{3.}$

Grading: +5 points for $\widetilde{V}$, *+5 points for the RREF, +5 points for the dimension. Grading for common mistakes: +7 points (total) for finding a basis for the null space; −3 points for using the columns of the RREF.*

MAT 242 Test 2 SOLUTIONS, FORM A

5. The eigenvalues of the matrix $A = \begin{bmatrix} 1 & -2 & -4 \\ 8 & 11 & 16 \\ -2 & -2 & -1 \end{bmatrix}$ are 3 (with multiplicity 2) and 5 (with multiplicity 1). (You do not need to find these.) Do the following for the matrix A:

   a. [10 points] Find a basis for the eigenspace of **each** eigenvalue.

Solution: The eigenspace of an eigenvalue λ is the null space of $A - \lambda I$. So, if $\lambda = 3$,

$$A - \lambda I = \begin{bmatrix} -2 & -2 & -4 \\ 8 & 8 & 16 \\ -2 & -2 & -4 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is parameterized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\alpha - 2\beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \beta \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Thus, $\boxed{\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}}$ *is a basis for the eigenspace of $\lambda = 3$.*

If $\lambda = 5$,

$$A - \lambda I = \begin{bmatrix} -4 & -2 & -4 \\ 8 & 6 & 16 \\ -2 & -2 & -6 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\boxed{\left\{ \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right\}}$ *is a basis for the eigenspace of $\lambda = 5$.*

Grading: +3 points for $A - \lambda I$, +3 points for the RREF, +4 points for finding the null space basis.

   b. [10 points] Is the matrix A diagonalizable? If so, find matrices D and P such that $A = PDP^{-1}$ and D is a diagonal matrix. If A is not diagonalizable explain carefully why it is not.

Solution: $\boxed{\text{YES.}}$ *The dimension of each eigenspace equals the (given) multiplicity of each eigenvalue. One pair of matrices that diagonalizes A is*

$$\boxed{D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} -2 & -1 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix}.}$$

Grading: +3 points for yes/no, +7 points for the explanation. Full credit — +10 points — was given for an answer consistent with any mistakes made in part* (a).

MAT 242 Test 2 SOLUTIONS, FORM B

1. [30 points] For the matrix A below, find a basis for the null space of A, a basis for the row space of A, a basis for the column space of A, the rank of A, and the nullity of A. The reduced row echelon form of A is the matrix R given below.

$$A = \begin{bmatrix} 0 & 3 & -3 & 9 & 3 & -6 \\ 2 & 3 & 2 & 4 & -27 & 4 \\ 5 & 5 & 5 & 10 & -60 & 10 \\ 4 & 0 & 3 & 11 & -32 & 6 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 0 & 5 & -5 & 0 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & -3 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: To find a basis for the null space, you need to solve the system of linear equations $A\vec{x} = \vec{0}$, or equivalently $R\vec{x} = \vec{0}$. Parameterizing the solutions to this equation produces

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -5\alpha + 5\beta \\ 3\beta \\ 3\alpha + 4\beta - 2\gamma \\ \alpha \\ \beta \\ \gamma \end{bmatrix} = \alpha \cdot \begin{bmatrix} -5 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 5 \\ 3 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so $\boxed{\left\{ \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}}$ *is a basis for the* **null space**. *The* **nullity** *is the number of vectors in this basis, which is* $\boxed{3.}$

A basis for the **row space** *can be found by taking the nonzero rows of R:*

$$\boxed{[1,\ 0,\ 0,\ 5,\ -5,\ 0], [0,\ 1,\ 0,\ 0,\ -3,\ 0], [0,\ 0,\ 1,\ -3,\ -4,\ 2]}$$

A basis for the **column space** *can be found by taking the columns of A which have pivots in them, so* $\boxed{\left\{ \begin{bmatrix} 0 \\ 2 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ 3 \end{bmatrix} \right\}}$ *is a basis for the column space.*

The **rank** *of A is the number of vectors in a basis for the row space (or column space) of A, so the rank of A is* $\boxed{3.}$

Grading: +10 points for finding a basis for the null space, +5 points for each of: a basis for the row space, a basis for the column space, the nullity, the rank. Grading for common mistakes: −3 points for forgetting a variable in the parameterization; −3 points for choosing columns of R for the column space of A; −3 points for choosing rows from A for the row space of A; −7 points for choosing the non-pivot columns of A for the null space of A.

MAT 242 Test 2 SOLUTIONS, FORM B

2. Let B be the (ordered) basis $\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\-1\\-1\end{bmatrix}, \begin{bmatrix}-1\\-1\\-2\end{bmatrix}\right)$ and C the basis $\left(\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}3\\-2\\6\end{bmatrix}, \begin{bmatrix}1\\-4\\1\end{bmatrix}\right)$.

a. [10 points] Find the coordinates of $\begin{bmatrix}14\\-50\\16\end{bmatrix}$ with respect to the basis C.

Solution:

$$[\vec{u}]_C = \widetilde{C}^{-1} \cdot \vec{u} = \begin{bmatrix}1 & 3 & 1\\-1 & -2 & -4\\2 & 6 & 1\end{bmatrix}^{-1} \cdot \begin{bmatrix}14\\-50\\16\end{bmatrix} = \boxed{\begin{bmatrix}2\\0\\12\end{bmatrix}}.$$

Grading: +3 points for the formula, +4 points for substitution, +3 points for calculation.

b. [10 points] If the coordinates of $\vec{u}$ with respect to C are $\begin{bmatrix}2\\4\\2\end{bmatrix}$, what are the coordinates of $\vec{u}$ with respect to B?

Solution:

$$[\vec{u}]_B = \widetilde{B}^{-1} \cdot \widetilde{C} \cdot [\vec{u}]_C = \begin{bmatrix}1 & -2 & -1\\0 & -1 & -1\\0 & -1 & -2\end{bmatrix}^{-1} \cdot \begin{bmatrix}1 & 3 & 1\\-1 & -2 & -4\\2 & 6 & 1\end{bmatrix} \cdot \begin{bmatrix}2\\4\\2\end{bmatrix} = \boxed{\begin{bmatrix}100\\66\\-48\end{bmatrix}}.$$

Grading: +3 points for the formula, +4 points for substitution, +3 points for calculation. Grading for common mistakes: +7 points for $\begin{bmatrix}114\\-38\\-8\end{bmatrix}$ *(backwards), +5 points for* $BC^{-1}[u] = \begin{bmatrix}-24\\12\\-50\end{bmatrix}$, *+5 points for* $CB^{-1}[u] = \begin{bmatrix}-62\\-14\\-16\end{bmatrix}$, *+5 points for* $B^{-1}[u]$, *+7 points for finding the change-of-basis matrix.*

MAT 242 Test 2 SOLUTIONS, FORM B

3. [15 points] Let $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ -3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 5 \\ 4 \\ -3 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -3 \\ 4 \\ -7 \\ 12 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 5 \\ 4 \\ 2 \\ -1 \end{bmatrix}$. Is the vector $\begin{bmatrix} 3 \\ 8 \\ 4 \\ -20 \end{bmatrix}$ in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$? Justify your answer.

Solution: You must determine whether the augmented matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4|\vec{u}]$ *is a system that has at least one solution.*

$$\left[\begin{array}{cccc|c} 2 & 5 & -3 & 5 & 3 \\ 0 & 4 & 4 & 4 & 8 \\ 1 & -3 & -7 & 2 & 4 \\ -3 & 0 & 12 & -1 & -20 \end{array}\right] \xrightarrow[RREF]{} \left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Since this system has no solutions, the vector $\boxed{\text{is not in the span.}}$

Grading: +4 points for setting up the matrix, +4 points for the RREF, +3 points for determining how many solutions there were, +4 points for answering YES/NO. Grading for common mistakes: −8 points for using $\vec{0}$ *instead of* $\vec{u}$.

4. [15 points] Find a basis for the subspace spanned by $\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\}$, and the dimension of that subspace.

Solution: To find this basis, use the column space (CS) approach: Glue the vectors together to get the matrix $\widetilde{V}$, *find the RREF, and take the original vectors that have pivots in their columns:*

$$\widetilde{V} = \begin{bmatrix} 1 & -3 & 2 & -1 & -4 \\ -1 & 3 & -2 & 3 & 0 \\ -2 & 6 & -4 & -1 & -3 \\ 1 & -3 & 2 & -1 & 0 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & -3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\boxed{\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\}}$ *is a basis for this subspace. Its dimension is the number of vectors in the basis, which is* $\boxed{3.}$

Grading: +5 points for $\widetilde{V}$, *+5 points for the RREF, +5 points for the dimension. Grading for common mistakes: +7 points (total) for finding a basis for the null space; −3 points for using the columns of the RREF.*

MAT 242 Test 2 SOLUTIONS, FORM B

5. The eigenvalues of the matrix $A = \begin{bmatrix} 3 & 30 & 60 \\ 0 & -15 & -36 \\ 0 & 6 & 15 \end{bmatrix}$ are 3 (with multiplicity 2) and -3 (with multiplicity 1). (You do not need to find these.) Do the following for the matrix A:

   a. [10 points] Find a basis for the eigenspace of **each** eigenvalue.

Solution: The eigenspace of an eigenvalue λ is the null space of $A - \lambda I$. So, if $\lambda = 3$,

$$A - \lambda I = \begin{bmatrix} 0 & 30 & 60 \\ 0 & -18 & -36 \\ 0 & 6 & 12 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is parameterized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ -2\beta \\ \beta \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Thus, $\boxed{\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$ *is a basis for the eigenspace of $\lambda = 3$.*

If $\lambda = -3$,

$$A - \lambda I = \begin{bmatrix} 6 & 30 & 60 \\ 0 & -12 & -36 \\ 0 & 6 & 18 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\boxed{\left\{ \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \right\}}$ *is a basis for the eigenspace of $\lambda = -3$.*

Grading: +3 points for $A - \lambda I$, +3 points for the RREF, +4 points for finding the null space basis.

   b. [10 points] Is the matrix A diagonalizable? If so, find matrices D and P such that $A = PDP^{-1}$ and D is a diagonal matrix. If A is not diagonalizable explain carefully why it is not.

Solution: $\boxed{\text{YES.}}$ *The dimension of each eigenspace equals the (given) multiplicity of each eigenvalue. One pair of matrices that diagonalizes A is*

$$\boxed{D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & 5 \\ -2 & 0 & -3 \\ 1 & 0 & 1 \end{bmatrix}.}$$

Grading: +3 points for yes/no, +7 points for the explanation. Full credit — +10 points — was given for an answer consistent with any mistakes made in part (a).*

MAT 242 Test 2 SOLUTIONS, FORM C

1. [30 points] For the matrix A below, find a basis for the null space of A, a basis for the row space of A, a basis for the column space of A, the rank of A, and the nullity of A. The reduced row echelon form of A is the matrix R given below.

$$A = \begin{bmatrix} -4 & -4 & 4 \\ 5 & 5 & -1 \\ 2 & 2 & -5 \\ 2 & 2 & -1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: To find a basis for the null space, you need to solve the system of linear equations $A\vec{x} = \vec{0}$, or equivalently $R\vec{x} = \vec{0}$. Parameterizing the solutions to this equation produces

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\alpha \\ \alpha \\ 0 \end{bmatrix} = \alpha \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

so $\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}}$ *is a basis for the* **null space**. *The* **nullity** *is the number of vectors in this basis, which is* $\boxed{1.}$

A basis for the **row space** *can be found by taking the nonzero rows of R:*

$$\boxed{\{[1,\ 1,\ 0], [0,\ 0,\ 1]\}}$$

A basis for the **column space** *can be found by taking the columns of A which have pivots in them, so* $\boxed{\left\{ \begin{bmatrix} -4 \\ 5 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ -5 \\ -1 \end{bmatrix} \right\}}$ *is a basis for the column space.*

The **rank** *of A is the number of vectors in a basis for the row space (or column space) of A, so the rank of A is* $\boxed{2.}$

Grading: +10 points for finding a basis for the null space, +5 points for each of: a basis for the row space, a basis for the column space, the nullity, the rank. Grading for common mistakes: −3 points for forgetting a variable in the parameterization; −3 points for choosing columns of R for the column space of A; −3 points for choosing rows from A for the row space of A; −7 points for choosing the non-pivot columns of A for the null space of A.

MAT 242 Test 2 SOLUTIONS, FORM C

2. Let B be the (ordered) basis $\left(\begin{bmatrix}1\\-3\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\-2\end{bmatrix}, \begin{bmatrix}-2\\9\\5\end{bmatrix}\right)$ and C the basis $\left(\begin{bmatrix}1\\3\\-3\end{bmatrix}, \begin{bmatrix}2\\7\\-6\end{bmatrix}, \begin{bmatrix}-3\\-8\\8\end{bmatrix}\right)$.

a. [10 points] Find the coordinates of $\begin{bmatrix}-2\\8\\2\end{bmatrix}$ with respect to the basis B.

Solution:

$$[\vec{u}]_B = \widetilde{B}^{-1} \cdot \vec{u} = \begin{bmatrix}1 & 0 & -2\\-3 & -1 & 9\\0 & -2 & 5\end{bmatrix}^{-1} \cdot \begin{bmatrix}-2\\8\\2\end{bmatrix} = \boxed{\begin{bmatrix}2\\4\\2\end{bmatrix}}.$$

Grading: +3 points for the formula, +4 points for substitution, +3 points for calculation.

b. [10 points] If the coordinates of $\vec{u}$ with respect to C are $\begin{bmatrix}1\\2\\3\end{bmatrix}$, what are the coordinates of $\vec{u}$ with respect to B?

Solution:

$$[\vec{u}]_B = \widetilde{B}^{-1} \cdot \widetilde{C} \cdot [\vec{u}]_C = \begin{bmatrix}1 & 0 & -2\\-3 & -1 & 9\\0 & -2 & 5\end{bmatrix}^{-1} \cdot \begin{bmatrix}1 & 2 & -3\\3 & 7 & -8\\-3 & -6 & 8\end{bmatrix} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = \boxed{\begin{bmatrix}-98\\-122\\-47\end{bmatrix}}.$$

Grading: +3 points for the formula, +4 points for substitution, +3 points for calculation. *Grading for common mistakes: +7 points for* $\begin{bmatrix}-59\\33\\4\end{bmatrix}$ *(backwards), +5 points for* $CB^{-1}[u] = \begin{bmatrix}26\\101\\-85\end{bmatrix}$, *+5 points for* $BC^{-1}[u] = \begin{bmatrix}-15\\22\\-40\end{bmatrix}$, *+5 points for* $B^{-1}[u]$, *+7 points for finding the change-of-basis matrix.*

MAT 242 Test 2 SOLUTIONS, FORM C

3. [15 points] Let $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ -3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -3 \\ 4 \\ 2 \\ 1 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} -21 \\ 5 \\ 2 \\ 4 \end{bmatrix}$. Is the vector $\begin{bmatrix} -22 \\ 10 \\ -11 \\ 3 \end{bmatrix}$ in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$? Justify your answer.

Solution: You must determine whether the augmented matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4|\vec{u}]$ *is a system that has at least one solution.*

$$\left[\begin{array}{rrrr|r} 2 & 2 & -3 & -21 & -22 \\ -1 & 1 & 4 & 5 & 10 \\ 4 & -5 & 2 & 2 & -11 \\ -3 & 3 & 1 & 4 & 3 \end{array}\right] \xrightarrow[RREF]{} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -5 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Since this system has no solutions, the vector $\boxed{\text{is not in the span.}}$

Grading: +4 points for setting up the matrix, +4 points for the RREF, +3 points for determining how many solutions there were, +4 points for answering YES/NO. Grading for common mistakes: −8 points for using $\vec{0}$ instead of $\vec{u}$.

4. [15 points] Find a basis for the subspace spanned by $\left\{ \begin{bmatrix} 3 \\ 4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 12 \\ 3 \\ 9 \\ 18 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \\ 1 \\ 3 \end{bmatrix} \right\}$, and the dimension of that subspace.

Solution: To find this basis, use the column space (CS) approach: Glue the vectors together to get the matrix $\widetilde{V}$, find the RREF, and take the original vectors that have pivots in their columns:

$$\widetilde{V} = \begin{bmatrix} 3 & 1 & -3 & 12 & -4 \\ 4 & 2 & 173 & -4 & \\ 3 & 0 & -3 & 9 & 1 \\ 1 & 3 & -3 & 18 & 3 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, $\boxed{\left\{ \begin{bmatrix} 3 \\ 4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \\ 1 \\ 3 \end{bmatrix} \right\}}$ *is a basis for this subspace. Its dimension is the number of vectors in the basis, which is* $\boxed{4.}$

Grading: +5 points for $\widetilde{V}$, +5 points for the RREF, +5 points for the dimension. Grading for common mistakes: +7 points (total) for finding a basis for the null space; −3 points for using the columns of the RREF.

MAT 242 Test 2 SOLUTIONS, FORM C

5. The eigenvalues of the matrix $A = \begin{bmatrix} 1 & -6 & -6 \\ -1 & -2 & -3 \\ 1 & 5 & 6 \end{bmatrix}$ are 1 (with multiplicity 2) and 3 (with multiplicity 1). (You do not need to find these.) Do the following for the matrix A:

   a. [10 points] Find a basis for the eigenspace of **each** eigenvalue.

Solution: The eigenspace of an eigenvalue λ is the null space of $A - \lambda I$. So, if $\lambda = 1$,

$$A - \lambda I = \begin{bmatrix} 0 & -6 & -6 \\ -1 & -3 & -3 \\ 1 & 5 & 5 \end{bmatrix} \xrightarrow[RREF]{🖩} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is parameterized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\alpha \\ \alpha \end{bmatrix} = \alpha \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Thus, $\boxed{\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}}$ *is a basis for the eigenspace of $\lambda = 1$.*

If $\lambda = 3$,

$$A - \lambda I = \begin{bmatrix} -2 & -6 & -6 \\ -1 & -5 & -3 \\ 1 & 5 & 3 \end{bmatrix} \xrightarrow[RREF]{🖩} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\boxed{\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}}$ *is a basis for the eigenspace of $\lambda = 3$.*

Grading: +3 points for $A - \lambda I$, +3 points for the RREF, +4 points for finding the null space basis.

   b. [10 points] Is the matrix A diagonalizable? If so, find matrices D and P such that $A = PDP^{-1}$ and D is a diagonal matrix. If A is not diagonalizable explain carefully why it is not.

Solution: $\boxed{\text{NO.}}$ *The dimension of the eigenspace of $\lambda = 1$ is less than its multiplicity ($1 < 2$).*

You could also have said that the matrix $P = \begin{bmatrix} 0 & -3 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}$ doesn't end up being square.

Grading: +3 points for yes/no, +7 points for the explanation. Full credit — +10 points — was given for an answer consistent with any mistakes made in part (a).*

MAT 242 Test 2 SOLUTIONS, FORM D

1. [30 points] For the matrix A below, find a basis for the null space of A, a basis for the row space of A, a basis for the column space of A, the rank of A, and the nullity of A. The reduced row echelon form of A is the matrix R given below.

$$A = \begin{bmatrix} -3 & 15 & -1 & 11 \\ 3 & -15 & -5 & -35 \\ -2 & 10 & 1 & 14 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -5 & 0 & -5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: To find a basis for the null space, you need to solve the system of linear equations $A\vec{x} = \vec{0}$, *or equivalently* $R\vec{x} = \vec{0}$. *Parameterizing the solutions to this equation produces*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5\alpha + 5\beta \\ \alpha \\ -4\beta \\ \beta \end{bmatrix} = \alpha \cdot \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 5 \\ 0 \\ -4 \\ 1 \end{bmatrix},$$

so $\boxed{\left\{ \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}}$ *is a basis for the* **null space**. *The* **nullity** *is the number of vectors in this basis, which is* $\boxed{2.}$

A basis for the **row space** *can be found by taking the nonzero rows of* R:

$$\boxed{\{[1,\ -5,\ 0,\ -5], [0,\ 0,\ 1,\ 4]\}}$$

A basis for the **column space** *can be found by taking the columns of* A *which have pivots in them, so* $\boxed{\left\{ \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} \right\}}$ *is a basis for the column space.*

The **rank** *of* A *is the number of vectors in a basis for the row space (or column space) of* A, *so the rank of* A *is* $\boxed{2.}$

Grading: +10 points for finding a basis for the null space, +5 points for each of: a basis for the row space, a basis for the column space, the nullity, the rank. Grading for common mistakes: -3 *points for forgetting a variable in the parameterization;* -3 *points for choosing columns of* R *for the column space of* A; -3 *points for choosing rows from* A *for the row space of* A; -7 *points for choosing the non-pivot columns of* A *for the null space of* A.

MAT 242 Test 2 SOLUTIONS, FORM D

2. Let B be the (ordered) basis $\left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix} \right)$ and C the basis $\left(\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \right)$.

   a. [10 points] Find the coordinates of $\begin{bmatrix} -12 \\ -22 \\ 16 \end{bmatrix}$ with respect to the basis B.

Solution:

$$[\vec{u}]_B = \widetilde{B}^{-1} \cdot \vec{u} = \begin{bmatrix} -1 & -3 & 1 \\ -2 & -5 & 1 \\ 1 & 6 & -5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -12 \\ -22 \\ 16 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}}.$$

Grading: +3 points for the formula, +4 points for substitution, +3 points for calculation.

   b. [10 points] If the coordinates of $\vec{u}$ with respect to B are $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, what are the coordinates of $\vec{u}$ with respect to C?

Solution:

$$[\vec{u}]_C = \widetilde{C}^{-1} \cdot \widetilde{B} \cdot [\vec{u}]_B = \begin{bmatrix} 1 & 3 & 1 \\ -1 & -4 & -3 \\ -1 & -3 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -1 & -3 & 1 \\ -2 & -5 & 1 \\ 1 & 6 & -5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} -73 \\ 25 \\ -6 \end{bmatrix}}.$$

Grading: +3 points for the formula, +4 points for substitution, +3 points for calculation. Grading for common mistakes: +7 points for $\begin{bmatrix} 338 \\ -155 \\ -117 \end{bmatrix}$ *(backwards), +5 points for* $CB^{-1}[u] = \begin{bmatrix} -9 \\ 21 \\ 5 \end{bmatrix}$, *+5 points for* $BC^{-1}[u] = \begin{bmatrix} 7 \\ -1 \\ -56 \end{bmatrix}$, *+5 points for* $C^{-1}[u] = \begin{bmatrix} 30 \\ -11 \\ 4 \end{bmatrix}$, *+7 points for finding the change-of-basis matrix.*

MAT 242 Test 2 SOLUTIONS, FORM D

3. [15 points] Let $\vec{v}_1 = \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -12 \\ 9 \\ 3 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 2 \\ 3 \\ -3 \\ -1 \end{bmatrix}$. Is the vector $\begin{bmatrix} 10 \\ -7 \\ 0 \\ 1 \end{bmatrix}$ in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$? Justify your answer.

Solution: You must determine whether the augmented matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4 | \vec{u}]$ *is a system that has at least one solution.*

$$\left[\begin{array}{cccc|c} -4 & -12 & 4 & 2 & 10 \\ 3 & 9 & 2 & 3 & -7 \\ 1 & 3 & -1 & -3 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right] \xrightarrow[RREF]{} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since this system has infinitely many solutions, it has at least one, and so the vector $\boxed{\text{is in the span.}}$

Grading: +4 points for setting up the matrix, +4 points for the RREF, +3 points for determining how many solutions there were, +4 points for answering YES/NO. Grading for common mistakes: −8 points for using $\vec{0}$ *instead of* $\vec{u}$.

4. [15 points] Find a basis for the subspace spanned by $\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -9 \\ -12 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$, and the dimension of that subspace.

Solution: To find this basis, use the column space (CS) approach: Glue the vectors together to get the matrix $\widetilde{V}$*, find the RREF, and take the original vectors that have pivots in their columns:*

$$\widetilde{V} = \begin{bmatrix} 1 & -3 & 3 & 2 & 4 \\ 3 & -9 & 1 & -4 & 0 \\ 4 & -12 & -3 & -1 & 3 \\ 1 & -3 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, $\boxed{\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}}$ *is a basis for this subspace. Its dimension is the number of vectors in the basis, which is* $\boxed{4.}$

Grading: +5 points for $\widetilde{V}$*, +5 points for the RREF, +5 points for the dimension. Grading for common mistakes: +7 points (total) for finding a basis for the null space; −3 points for using the columns of the RREF.*

MAT 242 Test 2 SOLUTIONS, FORM D

5. The eigenvalues of the matrix $A = \begin{bmatrix} -1 & 4 & 16 \\ 0 & 4 & 20 \\ 0 & -1 & -5 \end{bmatrix}$ are -1 (with multiplicity 2) and 0 (with multiplicity 1). (You do not need to find these.) Do the following for the matrix A:

   a. [10 points] Find a basis for the eigenspace of **each** eigenvalue.

Solution: The eigenspace of an eigenvalue λ is the null space of $A - \lambda I$. So, if $\lambda = -1$,

$$A - \lambda I = \begin{bmatrix} 0 & 4 & 16 \\ 0 & 5 & 20 \\ 0 & -1 & -4 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is parameterized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ -4\beta \\ \beta \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix}$$

Thus, $\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix} \right\}}$ *is a basis for the eigenspace of $\lambda = -1$.*

If $\lambda = 0$,

$$A - \lambda I = \begin{bmatrix} -1 & 4 & 16 \\ 0 & 4 & 20 \\ 0 & -1 & -5 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\boxed{\left\{ \begin{bmatrix} -4 \\ -5 \\ 1 \end{bmatrix} \right\}}$ *is a basis for the eigenspace of $\lambda = 0$.*

Grading: +3 points for $A - \lambda I$, +3 points for the RREF, +4 points for finding the null space basis.

   b. [10 points] Is the matrix A diagonalizable? If so, find matrices D and P such that $A = PDP^{-1}$ and D is a diagonal matrix. If A is not diagonalizable explain carefully why it is not.

Solution: $\boxed{\text{YES.}}$ *The dimension of each eigenspace equals the (given) multiplicity of each eigenvalue. One pair of matrices that diagonalizes A is*

$$\boxed{D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & -4 \\ -4 & 0 & -5 \\ 1 & 0 & 1 \end{bmatrix}}.$$

Grading: +3 points for yes/no, +7 points for the explanation. Full credit — +10 points — was given for an answer consistent with any mistakes made in part* (a).

MAT 242 Test 2 SOLUTIONS, FORM E

1. [30 points] For the matrix A below, find a basis for the null space of A, a basis for the row space of A, a basis for the column space of A, the rank of A, and the nullity of A. The reduced row echelon form of A is the matrix R given below.

$$A = \begin{bmatrix} 5 & -4 & -22 & -5 & 2 & 21 \\ 2 & -4 & -16 & 10 & -3 & 6 \\ 5 & 1 & -7 & -30 & 3 & 26 \\ 3 & -5 & -21 & 10 & -5 & 10 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & -2 & -5 & 0 & 5 \\ 0 & 1 & 3 & -5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: To find a basis for the null space, you need to solve the system of linear equations $A\vec{x} = \vec{0}$, or equivalently $R\vec{x} = \vec{0}$. Parameterizing the solutions to this equation produces

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2\alpha + 5\beta - 5\gamma \\ -3\alpha + 5\beta - \gamma \\ \alpha \\ \beta \\ 0 \\ \gamma \end{bmatrix} = \alpha \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 5 \\ 5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} -5 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so $\boxed{\left\{ \begin{bmatrix} -5 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$ *is a basis for the* **null space**. *The* **nullity** *is the number of vectors in this basis, which is* $\boxed{3.}$

A basis for the **row space** *can be found by taking the nonzero rows of R:*

$$\boxed{\{[1,\ 0,\ -2,\ -5,\ 0,\ 5], [0,\ 1,\ 3,\ -5,\ 0,\ 1], [0,\ 0,\ 0,\ 0,\ 1,\ 0]\}}$$

A basis for the **column space** *can be found by taking the columns of A which have pivots in them, so* $\boxed{\left\{ \begin{bmatrix} 5 \\ 2 \\ 5 \\ 3 \end{bmatrix} \begin{bmatrix} -4 \\ -4 \\ 1 \\ -5 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 3 \\ -5 \end{bmatrix} \right\}}$ *is a basis for the column space.*

The **rank** *of A is the number of vectors in a basis for the row space (or column space) of A, so the rank of A is* $\boxed{3.}$

Grading: +10 points for finding a basis for the null space, +5 points for each of: a basis for the row space, a basis for the column space, the nullity, the rank. Grading for common mistakes: −3 points for forgetting a variable in the parameterization; −3 points for choosing columns of R for the column space of A; −3 points for choosing rows from A for the row space of A; −7 points for choosing the non-pivot columns of A for the null space of A.

MAT 242 Test 2 SOLUTIONS, FORM E

2. Let B be the (ordered) basis $\left(\begin{bmatrix}-1\\1\\-2\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\-2\end{bmatrix}\right)$ and C the basis $\left(\begin{bmatrix}1\\-3\\1\end{bmatrix}, \begin{bmatrix}-2\\7\\1\end{bmatrix}, \begin{bmatrix}2\\-4\\7\end{bmatrix}\right)$.

a. [10 points] Find the coordinates of $\begin{bmatrix}-4\\6\\-4\end{bmatrix}$ with respect to the basis B.

Solution:

$$[\vec{u}]_B = \widetilde{B}^{-1} \cdot \vec{u} = \begin{bmatrix}-1 & 1 & -2\\-1 & 2 & -3\\-3 & 5 & -7\end{bmatrix}^{-1} \cdot \begin{bmatrix}-4\\6\\-4\end{bmatrix} = \boxed{\begin{bmatrix}2\\4\\2\end{bmatrix}}.$$

Grading: +3 points for the formula, +4 points for substitution, +3 points for calculation.

b. [10 points] If the coordinates of $\vec{u}$ with respect to B are $\begin{bmatrix}2\\4\\2\end{bmatrix}$, what are the coordinates of $\vec{u}$ with respect to C?

Solution:

$$[\vec{u}]_C = \widetilde{C}^{-1} \cdot \widetilde{B} \cdot [\vec{u}]_B = \begin{bmatrix}1 & -2 & 2\\-3 & 7 & -4\\1 & 1 & 7\end{bmatrix}^{-1} \cdot \begin{bmatrix}-1 & -1 & 1\\1 & 2 & -2\\-2 & 1 & -2\end{bmatrix} \cdot \begin{bmatrix}2\\4\\2\end{bmatrix} = \boxed{\begin{bmatrix}92\\30\\-18\end{bmatrix}}.$$

Grading: +3 points for the formula, +4 points for substitution, +3 points for calculation. Grading for common mistakes: +7 points for $\begin{bmatrix}-10\\24\\12\end{bmatrix}$ *(backwards), +5 points for* $CB^{-1}[u] = \begin{bmatrix}-20\\126\\158\end{bmatrix}$, *+5 points for* $BC^{-1}[u] = \begin{bmatrix}238\\-318\\206\end{bmatrix}$, *+5 points for* $C^{-1}[u] = \begin{bmatrix}6\\-1\\4\end{bmatrix}$, *+7 points for finding the change-of-basis matrix.*

MAT 242 Test 2 SOLUTIONS, FORM E

3. [15 points] Let $\vec{v}_1 = \begin{bmatrix} -4 \\ 2 \\ 5 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ -4 \\ 2 \\ -3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -3 \\ 5 \\ -1 \\ -1 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 2 \\ -8 \\ 7 \\ 1 \end{bmatrix}$. Is the vector $\begin{bmatrix} -11 \\ -14 \\ 6 \\ 1 \end{bmatrix}$ in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$? Justify your answer.

Solution: You must determine whether the augmented matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4|\vec{u}]$ is a system that has at least one solution.

$$\left[\begin{array}{rrrr|r} -4 & 3 & -3 & 2 & -11 \\ 2 & -4 & 5 & -8 & -14 \\ 5 & 2 & -1 & 7 & 6 \\ -1 & -3 & -1 & 1 & 1 \end{array}\right] \xrightarrow[RREF]{} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Since this system has no solutions, it has at least one, and so the vector $\boxed{\text{is not in the span.}}$

Grading: +4 points for setting up the matrix, +4 points for the RREF, +3 points for determining how many solutions there were, +4 points for answering YES/NO. Grading for common mistakes: −8 points for using $\vec{0}$ instead of $\vec{u}$.

4. [15 points] Find a basis for the subspace spanned by $\left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 9 \\ -12 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$, and the dimension of that subspace.

Solution: To find this basis, use the column space (CS) approach: Glue the vectors together to get the matrix $\widetilde{V}$, find the RREF, and take the original vectors that have pivots in their columns:

$$\widetilde{V} = \begin{bmatrix} 0 & -3 & 9 & -2 & 3 \\ -3 & 1 & -12 & -3 & 1 \\ 1 & 1 & 0 & 4 & 0 \\ -1 & -1 & 0 & -2 & 2 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, $\boxed{\left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}}$ *is a basis for this subspace. Its dimension is the number of vectors in the basis, which is* $\boxed{4.}$

Grading: +5 points for $\widetilde{V}$, +5 points for the RREF, +5 points for the dimension. Grading for common mistakes: +7 points (total) for finding a basis for the null space; −3 points for using the columns of the RREF.

MAT 242 Test 2 SOLUTIONS, FORM E

5. The eigenvalues of the matrix $A = \begin{bmatrix} -5 & -8 & -16 \\ -2 & -15 & -20 \\ 1 & 9 & 13 \end{bmatrix}$ are -5 (with multiplicity 2) and 3 (with multiplicity 1). (You do not need to find these.) Do the following for the matrix A:

   a. [10 points] Find a basis for the eigenspace of **each** eigenvalue.

Solution: The eigenspace of an eigenvalue λ is the null space of $A - \lambda I$. So, if $\lambda = -5$,

$$A - \lambda I = \begin{bmatrix} 0 & -8 & -16 \\ -2 & -10 & -20 \\ 1 & 9 & 18 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is parameterized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2\alpha \\ \alpha \end{bmatrix} = \alpha \cdot \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Thus, $\boxed{\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}}$ *is a basis for the eigenspace of $\lambda = -5$.*

If $\lambda = 3$,

$$A - \lambda I = \begin{bmatrix} -8 & -8 & -16 \\ -2 & -18 & -20 \\ 1 & 9 & 10 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\boxed{\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}}$ *is a basis for the eigenspace of $\lambda = 3$.*

Grading: +3 points for $A - \lambda I$, +3 points for the RREF, +4 points for finding the null space basis.

   b. [10 points] Is the matrix A diagonalizable? If so, find matrices D and P such that $A = PDP^{-1}$ and D is a diagonal matrix. If A is not diagonalizable explain carefully why it is not.

Solution: $\boxed{\text{NO.}}$ *The dimension of the eigenspace of $\lambda = -5$ is less than its multiplicity ($1 < 2$).*

You could also have said that the matrix $P = \begin{bmatrix} 0 & -1 \\ -2 & -1 \\ 1 & 1 \end{bmatrix}$ doesn't end up being square.

Grading: +3 points for yes/no, +7 points for the explanation. Full credit — $+10^*$ points — was given for an answer consistent with any mistakes made in part (a).

MAT 242 Test 2 SOLUTIONS, FORM F

1. [30 points] For the matrix A below, find a basis for the null space of A, a basis for the row space of A, a basis for the column space of A, the rank of A, and the nullity of A. The reduced row echelon form of A is the matrix R given below.

$$A = \begin{bmatrix} 2 & -2 & -2 & -1 & 14 & 18 \\ -4 & 4 & -1 & 1 & -13 & -7 \\ 4 & -4 & -3 & 4 & 25 & 7 \\ 4 & -4 & -3 & 0 & 25 & 23 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -1 & 0 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & -3 & -5 \\ 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: To find a basis for the null space, you need to solve the system of linear equations $A\vec{x} = \vec{0}$, or equivalently $R\vec{x} = \vec{0}$. Parameterizing the solutions to this equation produces

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \alpha - 4\beta - 2\gamma \\ \alpha \\ 3\beta + 5\gamma \\ 4\gamma \\ \beta \\ \gamma \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} -2 \\ 0 \\ 5 \\ 4 \\ 0 \\ 1 \end{bmatrix},$$

so $\boxed{\left\{ \begin{bmatrix} -2 \\ 0 \\ 5 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}}$ *is a basis for the* **null space**. *The* **nullity** *is the number of vectors in this basis, which is* $\boxed{3.}$

A basis for the **row space** *can be found by taking the nonzero rows of R:*

$$\boxed{\{[1,\ -1,\ 0,\ 0,\ 4,\ 2], [0,\ 0,\ 1,\ 0,\ -3,\ -5], [0,\ 0,\ 0,\ 1,\ 0,\ -4]\}}$$

A basis for the **column space** *can be found by taking the columns of A which have pivots in them, so* $\boxed{\left\{ \begin{bmatrix} 2 \\ -4 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}}$ *is a basis for the column space.*

The **rank** *of A is the number of vectors in a basis for the row space (or column space) of A, so the rank of A is* $\boxed{3.}$

Grading: +10 points for finding a basis for the null space, +5 points for each of: a basis for the row space, a basis for the column space, the nullity, the rank. Grading for common mistakes: −3 points for forgetting a variable in the parameterization; −3 points for choosing columns of R for the column space of A; −3 points for choosing rows from A for the row space of A; −7 points for choosing the non-pivot columns of A for the null space of A.

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2. Let B be the (ordered) basis $\left(\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right)$ and C the basis $\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \right)$.

a. [10 points] Find the coordinates of $\begin{bmatrix} 12 \\ -6 \\ -10 \end{bmatrix}$ with respect to the basis B.

Solution:

$$[\vec{u}]_B = \widetilde{B}^{-1} \cdot \vec{u} = \begin{bmatrix} 1 & 3 & -1 \\ -1 & -2 & 2 \\ -2 & -4 & 5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 12 \\ -6 \\ -10 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}}.$$

Grading: +3 points for the formula, +4 points for substitution, +3 points for calculation.

b. [10 points] If the coordinates of $\vec{u}$ with respect to C are $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, what are the coordinates of $\vec{u}$ with respect to B?

Solution:

$$[\vec{u}]_B = \widetilde{B}^{-1} \cdot \widetilde{C} \cdot [\vec{u}]_C = \begin{bmatrix} 1 & 3 & -1 \\ -1 & -2 & 2 \\ -2 & -4 & 5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 5 \\ 2 & -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} -90 \\ 31 \\ -7 \end{bmatrix}}.$$

Grading: +3 points for the formula, +4 points for substitution, +3 points for calculation. Grading for common mistakes: +7 points for $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ *(backwards), +5 points for* $CB^{-1}[u] = \begin{bmatrix} -15 \\ -21 \\ -35 \end{bmatrix}$, *+5 points for* $BC^{-1}[u] = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, *+5 points for* $B^{-1}[u] = \begin{bmatrix} -12 \\ 4 \\ -1 \end{bmatrix}$, *+7 points for finding the change-of-basis matrix.*

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3. [15 points] Let $\vec{v}_1 = \begin{bmatrix} 5 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 3 \\ -4 \\ 4 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -25 \\ -16 \\ 19 \\ -21 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 5 \end{bmatrix}$. Is the vector $\begin{bmatrix} -22 \\ -19 \\ 23 \\ -19 \end{bmatrix}$ in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$? Justify your answer.

Solution: You must determine whether the augmented matrix $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4|\vec{u}]$ is a system that has at least one solution.

$$\left[\begin{array}{rrrr|r} 5 & 4 & -25 & 2 & -22 \\ 1 & 3 & -16 & -1 & -19 \\ 1 & -4 & 19 & -1 & 23 \\ 1 & 4 & -21 & 5 & -19 \end{array}\right] \xrightarrow[RREF]{} \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -5 & 0 & -6 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since this system has infinitely many solutions, it has at least one, and so the vector $\boxed{\text{is in the span.}}$

Grading: +4 points for setting up the matrix, +4 points for the RREF, +3 points for determining how many solutions there were, +4 points for answering YES/NO. Grading for common mistakes: −8 points for using $\vec{0}$ instead of $\vec{u}$.

4. [15 points] Find a basis for the subspace spanned by $\left\{ \begin{bmatrix} -3 \\ -3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 12 \\ 12 \\ 4 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 18 \\ 13 \\ -11 \end{bmatrix} \right\}$, and the dimension of that subspace.

Solution: To find this basis, use the column space (CS) approach: Glue the vectors together to get the matrix $\widetilde{V}$, find the RREF, and take the original vectors that have pivots in their columns:

$$\widetilde{V} = \begin{bmatrix} -3 & 3 & 12 & -3 & 3 \\ -3 & 3 & 12 & 2 & 18 \\ -1 & 1 & 4 & 3 & 13 \\ 2 & -2 & -8 & -1 & -11 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & -1 & -4 & 0 & -4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\boxed{\left\{ \begin{bmatrix} -3 \\ -3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \right\}}$ *is a basis for this subspace. Its dimension is the number of vectors in the basis, which is* $\boxed{2.}$

Grading: +5 points for $\widetilde{V}$, +5 points for the RREF, +5 points for the dimension. Grading for common mistakes: +7 points (total) for finding a basis for the null space; −3 points for using the columns of the RREF.

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5. The eigenvalues of the matrix $A = \begin{bmatrix} 0 & 11 & -12 \\ 0 & 3 & 0 \\ 1 & -4 & 7 \end{bmatrix}$ are 3 (with multiplicity 2) and 4 (with multiplicity 1). (You do not need to find these.) Do the following for the matrix A:

   a. [10 points] Find a basis for the eigenspace of **each** eigenvalue.

Solution: The eigenspace of an eigenvalue λ is the null space of $A - \lambda I$. So, if $\lambda = 3$,

$$A - \lambda I = \begin{bmatrix} -3 & 11 & -12 \\ 0 & 0 & 0 \\ 1 & -4 & 4 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This is parameterized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4\alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \cdot \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $\boxed{\left\{ \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}}$ *is a basis for the eigenspace of $\lambda = 3$.*

If $\lambda = 4$,

$$A - \lambda I = \begin{bmatrix} -4 & 11 & -12 \\ 0 & -1 & 0 \\ 1 & -4 & 3 \end{bmatrix} \xrightarrow[RREF]{} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\boxed{\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}}$ *is a basis for the eigenspace of $\lambda = 4$.*

Grading: +3 points for $A - \lambda I$, +3 points for the RREF, +4 points for finding the null space basis.

   b. [10 points] Is the matrix A diagonalizable? If so, find matrices D and P such that $A = PDP^{-1}$ and D is a diagonal matrix. If A is not diagonalizable explain carefully why it is not.

Solution: $\boxed{\text{NO.}}$ *The dimension of the eigenspace of $\lambda = 3$ is less than its multiplicity ($1 < 2$).*

You could also have said that the matrix $P = \begin{bmatrix} -4 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$ *doesn't end up being square.*

Grading: +3 points for yes/no, +7 points for the explanation. Full credit — +10 points — was given for an answer consistent with any mistakes made in part* (a).