MAT 242 Test 1 SOLUTIONS, FORM A

1. [15 points] Use Cramer's Rule to solve the following system of linear equations for x.

$$\begin{aligned} -3x - 2y &= -1 \\ 2x - 2y &= 3 \end{aligned}$$

Solution:

$$x = \frac{\begin{vmatrix} -1 & -2 \\ 3 & -2 \end{vmatrix}}{\begin{vmatrix} -3 & -2 \\ 2 & -2 \end{vmatrix}} = \frac{8}{10} = \boxed{\frac{4}{5}.}$$

Grading: +5 points for each determinant, +5 points for combining them. Grading for common mistakes: +5 points (total) for another method which gave the correct answer; +3 points (total) for another method that gave the wrong answer.

2. [15 points] How many solutions does each of the following systems of linear equations have? Circle the entries which led you to your conclusion.

(a) $$\left[\begin{array}{ccccc|c} 1 & \mathbf{-1} & 0 & \mathbf{-1} & 0 & -4 \\ 0 & \mathbf{0} & 1 & \mathbf{-3} & 0 & -3 \\ 0 & \mathbf{0} & 0 & \mathbf{0} & 1 & 2 \\ 0 & \mathbf{0} & 0 & \mathbf{0} & 0 & 0 \end{array}\right]$$ $\boxed{\text{infinitely many}}$

(b) $$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{array}\right]$$ $\boxed{\text{None}}$

(c) $$\left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 2 \\ 0 & \mathbf{1} & 0 & 3 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ $\boxed{\text{Exactly one}}$

Solution: Correct answers are given above, and the relevant entries are in **boldface**.

Grading: +3 points for the correct answer, +2 points for indicating the entries (or for a brief explanation). Grading for common errors: −3 points for a wrong answer, −1 points for not indicating the relevant entries.

3. [20 points] Find the inverse of the matrix $\begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ using Gauss-Jordan Elimination. (Other methods may result in the loss of points.)

Solution: Typical row operations are shown below.

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-①]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{② + 2① \\ ③ + ①}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[③ - ②]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{① + ③ \\ ② + 2③}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

The inverse is thus $\boxed{\begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}}$.

Grading: +5 points for the set-up, +10 points for the row operations, +5 points for indicating the answer. Grading for common mistakes: −7 points for bad row operations; −5 points for only doing Gaussian Elimination; +10 points (total) for using another method correctly; +5 points (total) for using another method incorrectly.

4. [15 points] Solve the system of linear equations

$$\begin{aligned} 60x_1 - 30x_2 - 11x_3 + 3x_4 &= 0 \\ 198x_1 - 104x_2 - 39x_3 + 12x_4 &= 2 \\ -52x_1 + 27x_2 + 10x_3 - 3x_4 &= 4 \\ 15x_1 - 8x_2 - 3x_3 + x_4 &= 2 \end{aligned}$$

using the fact that $\begin{bmatrix} 60 & -30 & -11 & 3 \\ 198 & -104 & -39 & 12 \\ -52 & 27 & 10 & -3 \\ 15 & -8 & -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 1 & 9 & 6 \\ -2 & -3 & -12 & 6 \\ 3 & -1 & 6 & 22 \end{bmatrix}$. (Other methods may result in the loss of points.)

Solution: Use the $X = A^{-1}B$ formula:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 60 & -30 & -11 & 3 \\ 198 & -104 & -39 & 12 \\ -52 & 27 & 10 & -3 \\ 15 & -8 & -3 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 1 & 9 & 6 \\ -2 & -3 & -12 & 6 \\ 3 & -1 & 6 & 22 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 50 \\ -42 \\ 66 \end{bmatrix}$$

Grading: +5 points for the $A^{-1}B$ formula, +5 points for substituting A^{-1}, +5 points for doing the multiplication. Grading for common mistakes: −5 points for AB; +7 points (total) for using another method and getting the correct answer.

5. [15 points] Parameterize the solutions to the system of linear equations whose matrix, in reduced row echelon form, is below. Assume that the original variables were x_1, x_2, x_3, x_4, and x_5.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & -2 \\ 0 & 1 & 3 & -3 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: The variables which do not have pivots in their columns are x_3, x_4, and x_5. They are free variables, so set $x_3 = r$, $x_4 = s$, and $x_5 = t$. The rows of the matrix represent the equations

$$\begin{aligned} x_1 + x_3 - x_4 &= -2 \\ x_2 + 3x_3 - 3x_4 + x_5 &= -6 \end{aligned} \quad \text{so} \quad \begin{aligned} x_1 &= -2 - x_3 + x_4 = -2 - r + s \\ x_2 &= -6 - 3x_3 + 3x_4 - x_5 = -6 - 3r + 3s - t \end{aligned}$$

so the parameterization is

$$\boxed{\begin{aligned} & x_1 = -2 - r + s \\ & x_2 = -6 - 3r + 3s - t \\ & x_3 = r \\ & x_4 = s \\ & x_5 = t \\ & \text{where } r, s, t \text{ can be any real numbers} \end{aligned}}$$

Grading: +5 points for finding the free variables, +5 points for finding the equations for the lead variables, +2 points for writing them together, +3 points for including the "where ... can be any real numbers" condition.

6. [20 points] Find $\begin{vmatrix} 1 & -1 & -2 & 2 \\ -3 & 3 & 0 & -3 \\ -2 & 3 & -3 & -2 \\ 0 & 2 & 0 & 0 \end{vmatrix}$

Solution: Two methods will be presented below. First, expansion by minors, followed by Sarrus's method. (∗ denotes a matrix whose entries are unimportant.)

$$\begin{vmatrix} 1 & -1 & -2 & 2 \\ -3 & 3 & 0 & -3 \\ -2 & 3 & -3 & -2 \\ 0 & 2 & 0 & 0 \end{vmatrix} \overset{}{\underset{EM\,:\,R4}{=\!=\!=}} 0 \cdot - |*| + 2 \cdot + \begin{vmatrix} 1 & -2 & 2 \\ -3 & 0 & -3 \\ -2 & -3 & -2 \end{vmatrix} + 0 \cdot - |*| + 0 \cdot + |*|$$

$$= 2 \cdot [(0) + (-12) + (18) - (0) - (9) - (-12)] = \boxed{18.}$$

Gaussian elimination:

$$\begin{vmatrix} 1 & -1 & -2 & 2 \\ -3 & 3 & 0 & -3 \\ -2 & 3 & -3 & -2 \\ 0 & 2 & 0 & 0 \end{vmatrix} \underset{\substack{②+3① \\ ③+2①}}{=\!=\!=} \begin{vmatrix} 1 & -1 & -2 & 2 \\ 0 & 0 & -6 & 3 \\ 0 & 1 & -7 & 2 \\ 0 & 2 & 0 & 0 \end{vmatrix} \underset{② \leftrightarrow ③}{=\!=\!=} (-1) \cdot \begin{vmatrix} 1 & -1 & -2 & 2 \\ 0 & 1 & -7 & 2 \\ 0 & 0 & -6 & 3 \\ 0 & 2 & 0 & 0 \end{vmatrix}$$

$$\underset{④-2②}{=\!=\!=} (-1) \cdot \begin{vmatrix} 1 & -1 & -2 & 2 \\ 0 & 1 & -7 & 2 \\ 0 & 0 & -6 & 3 \\ 0 & 0 & 14 & -4 \end{vmatrix} \underset{④+2③}{=\!=\!=} (-1) \cdot \begin{vmatrix} 1 & -1 & -2 & 2 \\ 0 & 1 & -7 & 2 \\ 0 & 0 & -6 & 3 \\ 0 & 0 & 2 & 2 \end{vmatrix}$$

$$\underset{③ \leftrightarrow ④}{=\!=\!=} \begin{vmatrix} 1 & -1 & -2 & 2 \\ 0 & 1 & -7 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -6 & 3 \end{vmatrix} \underset{④+3③}{=\!=\!=} \begin{vmatrix} 1 & -1 & -2 & 2 \\ 0 & 1 & -7 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 9 \end{vmatrix}$$

$$= (1 \cdot 1 \cdot 2 \cdot 9) = \boxed{18.}$$

Grading for common mistakes: −2 points for not following the $+-+-$ *pattern; −10 points for bad row operations; −5 points for not keeping track of the effects of row operations on the determinant; −5 points for bad planning.*

MAT 242 Test 1 SOLUTIONS, FORM B

1. [15 points] Use Cramer's Rule to solve the following system of linear equations for y.

$$\begin{aligned} 2x + 2y &= 2 \\ 3x - 3y &= 0 \end{aligned}$$

Solution:

$$y = \frac{\begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 3 & -3 \end{vmatrix}} = \frac{-6}{-12} = \boxed{\frac{1}{2}.}$$

Grading: +5 points for each determinant, +5 points for combining them. Grading for common mistakes: +5 points (total) for another method which gave the correct answer; +3 points (total) for another method that gave the wrong answer.

2. [15 points] How many solutions does each of the following systems of linear equations have? Circle the entries which led you to your conclusion.

(a) $\left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & -3 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & -2 \end{array}\right]$ $\boxed{\text{Exactly one}}$

(b) $\left[\begin{array}{ccc|c} 1 & \mathbf{1} & 0 & 2 \\ 0 & \mathbf{0} & 1 & -3 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \end{array}\right]$ $\boxed{\text{Infinitely many}}$

(c) $\left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & 2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-2} \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\boxed{\text{None}}$

Solution: Correct answers are given above, and the relevant entries are in **boldface**.

Grading: +3 points for the correct answer, +2 points for indicating the entries (or for a brief explanation). Grading for common errors: −3 points for a wrong answer, −1 points for not indicating the relevant entries.

3. [20 points] Find the inverse of the matrix $\begin{bmatrix} 3 & 3 & -1 \\ 1 & -2 & 1 \\ 0 & -2 & 1 \end{bmatrix}$ using Gauss-Jordan Elimination.

Solution: Typical row operations are shown below.

$$\left[\begin{array}{rrr|rrr} 3 & 3 & -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{①} - 2\text{②}]{} \left[\begin{array}{rrr|rrr} 1 & 7 & -3 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{②} - \text{①}]{} \left[\begin{array}{rrr|rrr} 1 & 7 & -3 & 1 & -2 & 0 \\ 0 & -9 & 4 & -1 & 3 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{②} - 5\text{③}]{} \left[\begin{array}{rrr|rrr} 1 & 7 & -3 & 1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 3 & -5 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{③} + 2\text{②}]{} \left[\begin{array}{rrr|rrr} 1 & 7 & -3 & 1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 3 & -5 \\ 0 & 0 & -1 & -2 & 6 & -9 \end{array}\right]$$

$$\xrightarrow[-\text{③}]{} \left[\begin{array}{rrr|rrr} 1 & 7 & -3 & 1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 3 & -5 \\ 0 & 0 & 1 & 2 & -6 & 9 \end{array}\right]$$

$$\xrightarrow[\substack{\text{①} + 3\text{③} \\ \text{②} + \text{③}}]{} \left[\begin{array}{rrr|rrr} 1 & 7 & 0 & 7 & -20 & 27 \\ 0 & 1 & 0 & 1 & -3 & 4 \\ 0 & 0 & 1 & 2 & -6 & 9 \end{array}\right]$$

$$\xrightarrow[\text{①} - 7\text{②}]{} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -3 & 4 \\ 0 & 0 & 1 & 2 & -6 & 9 \end{array}\right]$$

The inverse is thus $\boxed{\begin{bmatrix} 0 & 1 & -1 \\ 1 & -3 & 4 \\ 2 & -6 & 9 \end{bmatrix}}$.

Grading: +5 points for the set-up, +10 points for the row operations, +5 points for indicating the answer. Grading for common mistakes: −7 points for bad row operations; −5 points for only doing Gaussian Elimination; +10 points (total) for using another method correctly; +5 points (total) for using another method incorrectly.

4. [15 points] Solve the system of linear equations

$$\begin{array}{rcrcrcrcl} -8x_1 & + & 6x_2 & - & 7x_3 & - & 5x_4 & = & 2 \\ -4x_1 & + & 2x_2 & + & 3x_3 & & & = & 0 \\ 3x_1 & - & 2x_2 & & & + & x_4 & = & 1 \\ 5x_1 & - & 3x_2 & - & x_3 & + & x_4 & = & 3 \end{array}$$

using the fact that $\begin{bmatrix} -8 & 6 & -7 & -5 \\ -4 & 2 & 3 & 0 \\ 3 & -2 & 0 & 1 \\ 5 & -3 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -3 & -3 & -2 \\ -2 & -7 & -3 & -7 \\ 0 & 1 & -2 & 2 \\ -1 & -5 & 4 & -8 \end{bmatrix}$. (Other methods may result in the loss of points.)

Solution: Use the $X = A^{-1}B$ formula:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -8 & 6 & -7 & -5 \\ -4 & 2 & 3 & 0 \\ 3 & -2 & 0 & 1 \\ 5 & -3 & -1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 & -3 & -3 & -2 \\ -2 & -7 & -3 & -7 \\ 0 & 1 & -2 & 2 \\ -1 & -5 & 4 & -8 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -11 \\ -28 \\ 4 \\ -22 \end{bmatrix}$$

Grading: +5 points for the $A^{-1}B$ formula, +5 points for substituting A^{-1}, +5 points for doing the multiplication. Grading for common mistakes: −5 points for AB; +7 points (total) for using another method and getting the correct answer.

5. [15 points] Parameterize the solutions to the system of linear equations whose matrix, in reduced row echelon form, is below. Assume that the original variables were x_1, x_2, x_3, and x_4.

$$\left[\begin{array}{rrrr|r} 1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: The variables which do not have pivots in their columns are x_2 and x_4. They are free variables, so set $x_2 = r$ and $x_4 = s$. The rows of the matrix represent the equations

$$\begin{aligned} x_1 - x_2 + 2x_4 &= 1 \\ x_3 + 3x_4 &= 7 \end{aligned} \quad \text{so} \quad \begin{aligned} x_1 &= 1 + x_2 - 2x_4 = 1 + r - 2s \\ x_3 &= 7 - 3x_4 = 7 - 3s \end{aligned}$$

so the parameterization is

$$\boxed{\begin{aligned} &x_1 = 1 + r - 2s \\ &x_2 = r \\ &x_3 = 7 - 3s \\ &x_4 = s \\ &\text{where } r, s \text{ can be any real numbers} \end{aligned}}$$

Grading: +5 points for finding the free variables, +5 points for finding the equations for the lead variables, +2 points for writing them together, +3 points for including the "where … can be any real numbers" condition.

6. [20 points] Find $\begin{vmatrix} 1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 \\ -3 & 3 & 0 & 2 \\ -1 & 0 & 0 & 2 \end{vmatrix}$

Solution: Two methods will be presented below. First, expansion by minors, followed by Sarrus's method. (∗ denotes a matrix whose entries are unimportant.)

$$\begin{vmatrix} 1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 \\ -3 & 3 & 0 & 2 \\ -1 & 0 & 0 & 2 \end{vmatrix} \overset{}{\underset{EM:C3}{=\!=\!=\!=}} 2 \cdot + \begin{vmatrix} 2 & 3 & 1 \\ -3 & 3 & 2 \\ -1 & 0 & 2 \end{vmatrix} + 0 \cdot - |*| + 0 \cdot + |*| + 0 \cdot - |*|$$

$$= 2 \cdot [(12) + (-6) + (0) - (-3) - (0) - (-18)] = \boxed{54.}$$

Gaussian elimination:

$$\begin{vmatrix} 1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 \\ -3 & 3 & 0 & 2 \\ -1 & 0 & 0 & 2 \end{vmatrix} \underset{\substack{ⓒ2 - 2ⓒ1 \\ ③ + 3① \\ ④ + ①}}{=\!=\!=\!=} \begin{vmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 6 & 6 & 2 \\ 0 & 1 & 2 & 2 \end{vmatrix} \underset{\substack{③ - 6② \\ ④ - ②}}{=\!=\!=\!=} \begin{vmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 30 & -4 \\ 0 & 0 & 6 & 1 \end{vmatrix}$$

$$\underset{③ \leftrightarrow ④}{=\!=\!=\!=} (-1) \cdot \begin{vmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 30 & -4 \end{vmatrix} \underset{④ - 5③}{=\!=\!=\!=} (-1) \cdot \begin{vmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & -9 \end{vmatrix}$$

$$= (-1) \cdot (1 \cdot 1 \cdot 6 \cdot (-9)) = \boxed{54.}$$

Grading for common mistakes: −2 points for not following the + − + − pattern; −10 points for bad row operations; −5 points for not keeping track of the effects of row operations on the determinant; −5 points for bad planning.

MAT 242 Test 1 SOLUTIONS, FORM C

1. [15 points] Use Cramer's Rule to solve the following system of linear equations for x.

$$\begin{aligned} -2x - 3y &= 1 \\ -2x + y &= 3 \end{aligned}$$

Solution:

$$x = \frac{\begin{vmatrix} 1 & -3 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} -2 & -3 \\ -2 & 1 \end{vmatrix}} = \frac{-5}{4} = \boxed{-\frac{5}{4}.}$$

Grading: +5 points for each determinant, +5 points for combining them. Grading for common mistakes: +5 points (total) for another method which gave the correct answer; +3 points (total) for another method that gave the wrong answer.

2. [15 points] How many solutions does each of the following systems of linear equations have? Circle the entries which led you to your conclusion.

(a) $\left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 2 \\ 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\boxed{\text{Exactly one}}$

(b) $\left[\begin{array}{ccc|c} 1 & \mathbf{3} & 0 & 0 \\ 0 & \mathbf{0} & 1 & -1 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \end{array}\right]$ $\boxed{\text{infinitely many}}$

(c) $\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\boxed{\text{None}}$

Solution: Correct answers are given above, and the relevant entries are in **boldface**.

Grading: +3 points for the correct answer, +2 points for indicating the entries (or for a brief explanation). Grading for common errors: −3 points for a wrong answer, −1 points for not indicating the relevant entries.

3. [20 points] Find the inverse of the matrix $\begin{bmatrix} 0 & -3 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ using Gauss-Jordan Elimination.

Solution: Typical row operations are shown below.

$$\left[\begin{array}{ccc|ccc} 0 & -3 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{①} \leftrightarrow \text{③}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -3 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[\text{②} + \text{①}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & -3 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[\text{③} + 3\text{②}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 3 & 3 \end{array}\right]$$

$$\xrightarrow[-\text{③}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -3 & -3 \end{array}\right]$$

The inverse is thus $\boxed{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -3 & -3 \end{bmatrix}}$.

Grading: +5 *points for the set-up,* +10 *points for the row operations,* +5 *points for indicating the answer. Grading for common mistakes:* −7 *points for bad row operations;* −5 *points for only doing Gaussian Elimination;* +10 *points (total) for using another method correctly;* +5 *points (total) for using another method incorrectly.*

4. [15 points] Solve the system of linear equations

$$\begin{array}{rcrcrcrcl} -9x_1 & - & 2x_2 & - & 4x_3 & & & = & 2 \\ -10x_1 & - & 3x_2 & - & 2x_3 & - & x_4 & = & 0 \\ 4x_1 & + & x_2 & + & x_3 & & & = & 1 \\ -x_1 & - & x_2 & + & x_3 & - & x_4 & = & 3 \end{array}$$

using the fact that $\begin{bmatrix} -9 & -2 & -4 & 0 \\ -10 & -3 & -2 & -1 \\ 4 & 1 & 1 & 0 \\ -1 & -1 & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 & -2 & 2 \\ -3 & 7 & 9 & -7 \\ -1 & 1 & 0 & -1 \\ 1 & -4 & -7 & 3 \end{bmatrix}$. (Other methods may result in the loss of points.)

Solution: Use the $X = A^{-1}B$ formula:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9 & -2 & -4 & 0 \\ -10 & -3 & -2 & -1 \\ 4 & 1 & 1 & 0 \\ -1 & -1 & 1 & -1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 & 2 \\ -3 & 7 & 9 & -7 \\ -1 & 1 & 0 & -1 \\ 1 & -4 & -7 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -18 \\ -5 \\ 4 \end{bmatrix}$$

Grading: +5 points for the $A^{-1}B$ formula, +5 points for substituting A^{-1}, +5 points for doing the multiplication. Grading for common mistakes: −5 points for AB; +7 points (total) for using another method and getting the correct answer.

5. [15 points] Parameterize the solutions to the system of linear equations whose matrix, in reduced row echelon form, is below. Assume that the original variables were x_1, x_2, x_3, and x_4.

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 2 & 4 \\ 0 & 1 & -2 & 1 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: The variables which do not have pivots in their columns are x_3 and x_4. They are free variables, so set $x_3 = r$ and $x_4 = s$. The rows of the matrix represent the equations

$$\begin{aligned} x_1 + 2x_3 + 2x_4 &= 4 \\ x_2 - 2x_3 + x_4 &= -8 \end{aligned} \quad \text{so} \quad \begin{aligned} x_1 &= 4 - 2x_3 - 2x_4 = 4 - 2r - 2s \\ x_2 &= -8 + 2x_3 - x_4 = -8 + 2r - s \end{aligned}$$

so the parameterization is

$$\boxed{\begin{aligned} & x_1 = 4 - 2r - 2s \\ & x_2 = -8 + 2r - s \\ & x_3 = r \\ & x_4 = s \\ & \textit{where } r, s \textit{ can be any real numbers} \end{aligned}}$$

Grading: +5 points for finding the free variables, +5 points for finding the equations for the lead variables, +2 points for writing them together, +3 points for including the "where ... can be any real numbers" condition.

6. [20 points] Find $\begin{vmatrix} 0 & -1 & 3 & 0 \\ 0 & -3 & 2 & -3 \\ 3 & -1 & -3 & -3 \\ 0 & 1 & 0 & 3 \end{vmatrix}$

Solution: Two methods will be presented below. First, expansion by minors, followed by Sarrus's method. ($*$ *denotes a matrix whose entries are unimportant.)*

$$\begin{vmatrix} 0 & -1 & 3 & 0 \\ 0 & -3 & 2 & -3 \\ 3 & -1 & -3 & -3 \\ 0 & 1 & 0 & 3 \end{vmatrix} \xlongequal[EM:C1]{} 0 \cdot + |*| + 0 \cdot - |*| + 3 \cdot + \begin{vmatrix} -1 & 3 & 0 \\ -3 & 2 & -3 \\ 1 & 0 & 3 \end{vmatrix} + 0 \cdot - |*|$$

$$= 3 \cdot [(-6) + (-9) + (0) - (0) - (0) - (-27)] = \boxed{36.}$$

Gaussian elimination:

$$\begin{vmatrix} 0 & -1 & 3 & 0 \\ 0 & -3 & 2 & -3 \\ 3 & -1 & -3 & -3 \\ 0 & 1 & 0 & 3 \end{vmatrix} \xlongequal[\substack{① \leftrightarrow ③ \\ ② \leftrightarrow ④}]{} \begin{vmatrix} 3 & -1 & -3 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 3 & 0 \\ 0 & -3 & 2 & -3 \end{vmatrix} \xlongequal[\substack{③ + ② \\ ④ + 3②}]{} \begin{vmatrix} 3 & -1 & -3 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 2 & 6 \end{vmatrix}$$

$$\xlongequal[③ - ④]{} \begin{vmatrix} 3 & -1 & -3 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 2 & 6 \end{vmatrix} \xlongequal[④ - 2③]{} \begin{vmatrix} 3 & -1 & -3 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 12 \end{vmatrix}$$

$$= (3 \cdot 1 \cdot 1 \cdot 12) = \boxed{36.}$$

Grading for common mistakes: -2 *points for not following the* $+-+-$ *pattern;* -10 *points for bad row operations;* -5 *points for not keeping track of the effects of row operations on the determinant;* -5 *points for bad planning.*

MAT 242 Test 1 SOLUTIONS, FORM D

1. [15 points] Use Cramer's Rule to solve the following system of linear equations for y.

$$\begin{aligned} 3x - y &= 0 \\ -2x - 2y &= 3 \end{aligned}$$

Solution:

$$y = \frac{\begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ -2 & -2 \end{vmatrix}} = \frac{9}{-8} = \boxed{-\frac{9}{8}.}$$

Grading: +5 points for each determinant, +5 points for combining them. Grading for common mistakes: +5 points (total) for another method which gave the correct answer; +3 points (total) for another method that gave the wrong answer.

2. [15 points] How many solutions does each of the following systems of linear equations have? Circle the entries which led you to your conclusion.

(a) $\left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & -2 \\ 0 & \mathbf{1} & 0 & 3 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\boxed{\text{Exactly one}}$

(b) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\boxed{\text{None}}$

(c) $\left[\begin{array}{ccc|c} 1 & 0 & \mathbf{1} & 1 \\ 0 & 1 & \mathbf{2} & 0 \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & \mathbf{0} & 0 \end{array}\right]$ $\boxed{\text{infinitely many}}$

Solution: Correct answers are given above, and the relevant entries are in **boldface**.

Grading: +3 points for the correct answer, +2 points for indicating the entries (or for a brief explanation). Grading for common errors: −3 points for a wrong answer, −1 points for not indicating the relevant entries.

3. [20 points] Find the inverse of the matrix $\begin{bmatrix} 3 & 1 & -1 \\ 4 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$ using Gauss-Jordan Elimination.

Solution: Typical row operations are shown below.

$$\left[\begin{array}{ccc|ccc} 3 & 1 & -1 & 1 & 0 & 0 \\ 4 & 2 & -1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{①} - \text{③}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 4 & 2 & -1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{\text{②} - 4\text{①} \\ \text{③} - 2\text{①}}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 2 & -1 & -4 & 1 & 4 \\ 0 & 1 & -1 & -2 & 0 & 3 \end{array}\right]$$

$$\xrightarrow[\text{②} \leftrightarrow \text{③}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & -2 & 0 & 3 \\ 0 & 2 & -1 & -4 & 1 & 4 \end{array}\right]$$

$$\xrightarrow[\text{③} - 2\text{②}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow[\text{②} + \text{③}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{array}\right]$$

The inverse is thus $\boxed{\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}}$.

Grading: +5 points for the set-up, +10 points for the row operations, +5 points for indicating the answer. Grading for common mistakes: −7 points for bad row operations; −5 points for only doing Gaussian Elimination; +10 points (total) for using another method correctly; +5 points (total) for using another method incorrectly.

4. [15 points] Solve the system of linear equations

$$\begin{array}{rcrcrcrcl} 53x_1 &+& 10x_2 &-& 27x_3 &+& 11x_4 &=& 0 \\ -20x_1 &-& 3x_2 &+& 10x_3 &-& 4x_4 &=& 2 \\ 10x_1 &+& 2x_2 &-& 5x_3 &+& 2x_4 &=& 4 \\ -4x_1 &-& x_2 &+& 2x_3 &-& x_4 &=& 2 \end{array}$$

using the fact that $\begin{bmatrix} 53 & 10 & -27 & 11 \\ -20 & -3 & 10 & -4 \\ 10 & 2 & -5 & 2 \\ -4 & -1 & 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -1 & 3 & -1 \\ 0 & 1 & 2 & 0 \\ -2 & -2 & 5 & -4 \\ 0 & -1 & -4 & -5 \end{bmatrix}$. (Other methods may result in the loss of points.)

Solution: *Use the* $X = A^{-1}B$ *formula:*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 53 & 10 & -27 & 11 \\ -20 & -3 & 10 & -4 \\ 10 & 2 & -5 & 2 \\ -4 & -1 & 2 & -1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 & -1 \\ 0 & 1 & 2 & 0 \\ -2 & -2 & 5 & -4 \\ 0 & -1 & -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ 8 \\ -28 \end{bmatrix}$$

Grading: +5 points for the $A^{-1}B$ *formula, +5 points for substituting* A^{-1}*, +5 points for doing the multiplication. Grading for common mistakes: −5 points for* AB*; +7 points (total) for using another method and getting the correct answer.*

5. [15 points] Parameterize the solutions to the system of linear equations whose matrix, in reduced row echelon form, is below. Assume that the original variables were x_1, x_2, x_3, and x_4.

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 8 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: There is only one variable that does not have a pivot in its column: x_3*. It is a free variable, so set* $x_3 = s$*. The rows of the matrix represent the equations*

$$\begin{aligned} x_1 + 3x_3 &= 8 \\ x_2 + 2x_3 &= 5 \\ x_4 &= -3 \end{aligned} \qquad \text{so} \qquad \begin{aligned} x_1 &= 8 - 3x_3 = 8 - 3r \\ x_2 &= 5 - 2x_3 = 5 - 2r \end{aligned}$$

so the parameterization is

$$\boxed{\begin{aligned} &x_1 = 8 - 3r \\ &x_2 = 5 - 2r \\ &x_3 = r \\ &x_4 = -3 \\ &\text{where } r \text{ can be any real number} \end{aligned}}$$

Grading: +5 points for finding the free variables, +5 points for finding the equations for the lead variables, +2 points for writing them together, +3 points for including the "where . . . can be any real numbers" condition.

6. [20 points] Find $\begin{vmatrix} 0 & 2 & 0 & -1 \\ 2 & 1 & -2 & -3 \\ 0 & -3 & -1 & 3 \\ 2 & 0 & 0 & 0 \end{vmatrix}$

Solution: Two methods will be presented below. First, expansion by minors, followed by Sarrus's method. (∗ denotes a matrix whose entries are unimportant.)

$$\begin{vmatrix} 0 & 2 & 0 & -1 \\ 2 & 1 & -2 & -3 \\ 0 & -3 & -1 & 3 \\ 2 & 0 & 0 & 0 \end{vmatrix} \underset{EM:R4}{=\!=\!=\!=} 2 \cdot - \begin{vmatrix} 2 & 0 & -1 \\ 1 & -2 & -3 \\ -3 & -1 & 3 \end{vmatrix} + 0 \cdot + |*| + 0 \cdot - |*| + 0 \cdot + |*|$$

$$= -2[(-12) + (0) + (1) - (-6) - (0) - (6)] = \boxed{22.}$$

Gaussian elimination:

$$\begin{vmatrix} 0 & 2 & 0 & -1 \\ 2 & 1 & -2 & -3 \\ 0 & -3 & -1 & 3 \\ 2 & 0 & 0 & 0 \end{vmatrix} \underset{①\leftrightarrow④}{=\!=\!=\!=} (-1) \cdot \begin{vmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & -2 & -3 \\ 0 & -3 & -1 & 3 \\ 0 & 2 & 0 & -1 \end{vmatrix} \underset{②-①}{=\!=\!=\!=} (-1) \cdot \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & -1 & 3 \\ 0 & 2 & 0 & -1 \end{vmatrix}$$

$$\underset{\substack{③+3②\\ ④-2②}}{=\!=\!=\!=} (-1) \cdot \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -7 & -6 \\ 0 & 0 & 4 & 5 \end{vmatrix} \underset{③+2④}{=\!=\!=\!=} (-1) \cdot \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 4 & 5 \end{vmatrix}$$

$$\underset{④-4③}{=\!=\!=\!=} (-1) \cdot \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -11 \end{vmatrix} = (-1) \cdot (2 \cdot 1 \cdot 1 \cdot (-11)) = \boxed{22.}$$

Grading for common mistakes: −2 points for not following the + − + − pattern; −10 points for bad row operations; −5 points for not keeping track of the effects of row operations on the determinant; −5 points for bad planning.

MAT 242 Test 1 SOLUTIONS, FORM E

1. [15 points] Use Cramer's Rule to solve the following system of linear equations for x.

$$\begin{aligned} -2x - 3y &= 1 \\ 3x - 2y &= -2 \end{aligned}$$

Solution:

$$x = \frac{\begin{vmatrix} 1 & -3 \\ -2 & -2 \end{vmatrix}}{\begin{vmatrix} -2 & -3 \\ 3 & -2 \end{vmatrix}} = \frac{-8}{13} = \boxed{-\frac{8}{13}.}$$

Grading: +5 points for each determinant, +5 points for combining them. Grading for common mistakes: +5 points (total) for another method which gave the correct answer; +3 points (total) for another method that gave the wrong answer.

2. [15 points] How many solutions does each of the following systems of linear equations have? Circle the entries which led you to your conclusion.

(a) $\left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 2 \\ 0 & \mathbf{1} & 0 & -3 \\ 0 & 0 & \mathbf{1} & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\boxed{\text{Exactly one}}$

(b) $\left[\begin{array}{ccc|c} 1 & -\mathbf{3} & 0 & -5 \\ 0 & \mathbf{0} & 1 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \end{array}\right]$ $\boxed{\text{Infinitely many}}$

(c) $\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} \end{array}\right]$ $\boxed{\text{None}}$

Solution: Correct answers are given above, and the relevant entries are in **boldface**.

Grading: +3 points for the correct answer, +2 points for indicating the entries (or for a brief explanation). Grading for common errors: −3 points for a wrong answer, −1 points for not indicating the relevant entries.

3. [20 points] Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ using Gauss-Jordan Elimination.

Solution: Typical row operations are shown below.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{③} - \text{①}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-\text{②}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{①} + \text{③}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

The inverse is thus $\boxed{\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}$.

Grading: +5 points for the set-up, +10 points for the row operations, +5 points for indicating the answer. Grading for common mistakes: −7 points for bad row operations; −5 points for only doing Gaussian Elimination; +10 points (total) for using another method correctly; +5 points (total) for using another method incorrectly.

4. [15 points] Solve the system of linear equations

$$\begin{aligned} 354x_1 + 51x_2 - 77x_3 - 24x_4 &= 0 \\ -75x_1 - 11x_2 + 16x_3 + 5x_4 &= 2 \\ -44x_1 - 6x_2 + 10x_3 + 3x_4 &= 4 \\ -14x_1 - 2x_2 + 3x_3 + x_4 &= 2 \end{aligned}$$

using the fact that $\begin{bmatrix} 354 & 51 & -77 & -24 \\ -75 & -11 & 16 & 5 \\ -44 & -6 & 10 & 3 \\ -14 & -2 & 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -3 & -2 & -3 \\ 3 & 8 & 7 & 11 \\ -2 & -6 & -3 & -9 \\ -2 & -8 & -5 & 8 \end{bmatrix}$. (Other methods may result in the loss of points.)

Solution: Use the $X = A^{-1}B$ formula:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 354 & 51 & -77 & -24 \\ -75 & -11 & 16 & 5 \\ -44 & -6 & 10 & 3 \\ -14 & -2 & 3 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 & -3 & -2 & -3 \\ 3 & 8 & 7 & 11 \\ -2 & -6 & -3 & -9 \\ -2 & -8 & -5 & 8 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -20 \\ 66 \\ -42 \\ -20 \end{bmatrix}$$

Grading: +5 points for the $A^{-1}B$ formula, +5 points for substituting A^{-1}, +5 points for doing the multiplication. Grading for common mistakes: −5 points for AB; +5 points (total) for using another method and getting the correct answer.

5. [15 points] Parameterize the solutions to the system of linear equations whose matrix, in reduced row echelon form, is below. Assume that the original variables were x_1, x_2, x_3, and x_4.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: The variables which do not have pivots in their columns are x_3 and x_4. They are free variables, so set $x_3 = r$ and $x_4 = s$. The rows of the matrix represent the equations

$$\begin{aligned} x_1 + x_4 &= 3 \\ x_2 + 2x_3 + x_4 &= 2 \end{aligned} \quad \textit{so} \quad \begin{aligned} x_1 &= 3 - x_4 = 3 - s \\ x_2 &= 2 - 2x_3 + x_4 = 2 - 2r - s \end{aligned}$$

so the parameterization is

$$\boxed{\begin{aligned} & x_1 = 3 - s \\ & x_2 = 2 - 2r - s \\ & x_3 = r \\ & x_4 = s \\ & \textit{where } r, s \textit{ can be any real numbers} \end{aligned}}$$

Grading: +5 points for finding the free variables, +5 points for finding the equations for the lead variables, +2 points for writing them together, +3 points for including the "where … can be any real numbers" condition.

6. [20 points] Find $\begin{vmatrix} 0 & 1 & 1 & 2 \\ 0 & 2 & 0 & -2 \\ -3 & 0 & -1 & 3 \\ 0 & -3 & -2 & -2 \end{vmatrix}$

Solution: Two methods will be presented below. First, expansion by minors, followed by Sarrus's method. (∗ denotes a matrix whose entries are unimportant.)

$$\begin{vmatrix} 0 & 1 & 1 & 2 \\ 0 & 2 & 0 & -2 \\ -3 & 0 & -1 & 3 \\ 0 & -3 & -2 & -2 \end{vmatrix} \underset{EM:C1}{=\!=\!=\!=} 0 \cdot + |*| + 0 \cdot - |*| + (-3) \cdot + \begin{vmatrix} 1 & 1 & 2 \\ 2 & 0 & -2 \\ -3 & -2 & -2 \end{vmatrix} + 0 \cdot - |*|$$

$$= (-3)[(0) + (6) + (-8) - (0) - (4) - (-4)] = \boxed{6.}$$

Gaussian elimination:

$$\begin{vmatrix} 0 & 1 & 1 & 2 \\ 0 & 2 & 0 & -2 \\ -3 & 0 & -1 & 3 \\ 0 & -3 & -2 & -2 \end{vmatrix} \underset{①\leftrightarrow③}{=\!=\!=\!=} (-1) \cdot \begin{vmatrix} -3 & 0 & -1 & 3 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & -3 & -2 & -2 \end{vmatrix} \underset{②\leftrightarrow③}{=\!=\!=\!=} \begin{vmatrix} -3 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 0 & -2 \\ 0 & -3 & -2 & -2 \end{vmatrix}$$

$$\underset{\substack{③-2② \\ ④+3②}}{=\!=\!=\!=} \begin{vmatrix} -3 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & 1 & 4 \end{vmatrix} \underset{③\leftrightarrow④}{=\!=\!=\!=} (-1) \cdot \begin{vmatrix} -3 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -2 & -6 \end{vmatrix}$$

$$\underset{④+2③}{=\!=\!=\!=} (-1) \cdot \begin{vmatrix} -3 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{vmatrix} = (-1) \cdot (-3 \cdot 1 \cdot 1 \cdot 2) = \boxed{6.}$$

Grading for common mistakes: −2 points for not following the + − + − pattern; −10 points for bad row operations; −5 points for not keeping track of the effects of row operations on the determinant; −5 points for bad planning.

MAT 242 Test 1 SOLUTIONS, FORM F

1. [15 points] Use Cramer's Rule to solve the following system of linear equations for y.

$$\begin{aligned} 2x + 3y &= -2 \\ -3x - y &= 1 \end{aligned}$$

Solution:

$$y = \frac{\begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ -3 & -1 \end{vmatrix}} = \frac{-4}{7} = \boxed{-\frac{4}{7}.}$$

Grading: +5 points for each determinant, +5 points for combining them. Grading for common mistakes: +5 points (total) for another method which gave the correct answer; +3 points (total) for another method that gave the wrong answer.

2. [15 points] How many solutions does each of the following systems of linear equations have? Circle the entries which led you to your conclusion.

(a) $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{array}\right]$ $\boxed{\text{None}}$

(b) $\left[\begin{array}{cccc|c} \mathbf{1} & 0 & 0 & 0 & -2 \\ 0 & \mathbf{1} & 0 & 0 & 3 \\ 0 & 0 & \mathbf{1} & 0 & 2 \\ 0 & 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\boxed{\text{Exactly one}}$

(c) $\left[\begin{array}{ccc|c} 1 & \mathbf{3} & 0 & -6 \\ 0 & \mathbf{0} & 1 & -3 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \end{array}\right]$ $\boxed{\text{Infinitely many}}$

Solution: Correct answers are given above, and the relevant entries are in **boldface**.

Grading: +3 points for the correct answer, +2 points for indicating the entries (or for a brief explanation). Grading for common errors: −3 points for a wrong answer, −1 points for not indicating the relevant entries.

3. [20 points] Find the inverse of the matrix $\begin{bmatrix} -3 & 0 & 1 \\ -5 & 1 & 3 \\ -2 & 0 & 1 \end{bmatrix}$ using Gauss-Jordan Elimination.

Solution: Typical row operations are shown below.

$$\left[\begin{array}{ccc|ccc} -3 & 0 & 1 & 1 & 0 & 0 \\ -5 & 1 & 3 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{①} - 2\text{③}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -2 \\ -5 & 1 & 3 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{\text{②} + 5\text{①} \\ \text{③} + 2\text{①}}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & -2 & 5 & 1 & -10 \\ 0 & 0 & -1 & 2 & 0 & -3 \end{array}\right]$$

$$\xrightarrow[-\text{③}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & -2 & 5 & 1 & -10 \\ 0 & 0 & 1 & -2 & 0 & 3 \end{array}\right]$$

$$\xrightarrow[\substack{\text{①} + \text{③} \\ \text{②} + 2\text{③}}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & -2 & 0 & 3 \end{array}\right]$$

The inverse is thus $\boxed{\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & -4 \\ -2 & 0 & 3 \end{bmatrix}}$.

Grading: +5 points for the set-up, +10 points for the row operations, +5 points for indicating the answer. Grading for common mistakes: −7 points for bad row operations; −5 points for only doing Gaussian Elimination; +10 points (total) for using another method correctly; +5 points (total) for using another method incorrectly.

4. [15 points] Solve the system of linear equations

$$\begin{aligned} 28x_1 - 6x_2 - 2x_3 - x_4 &= 0 \\ 105x_1 - 20x_2 - 9x_3 - 6x_4 &= 2 \\ 60x_1 - 12x_2 - 5x_3 - 3x_4 &= 4 \\ -24x_1 + 5x_2 + 2x_3 + x_4 &= 2 \end{aligned}$$

using the fact that $\begin{bmatrix} 28 & -6 & -2 & -1 \\ 105 & -20 & -9 & -6 \\ 60 & -12 & -5 & -3 \\ -24 & 5 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & -3 & -2 \\ 3 & 4 & -12 & -9 \\ 3 & 0 & 1 & 6 \\ 3 & 4 & -14 & -14 \end{bmatrix}$. (Other methods may result in the loss of points.)

Solution: Use the $X = A^{-1}B$ formula:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 28 & -6 & -2 & -1 \\ 105 & -20 & -9 & -6 \\ 60 & -12 & -5 & -3 \\ -24 & 5 & 2 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -3 & -2 \\ 3 & 4 & -12 & -9 \\ 3 & 0 & 1 & 6 \\ 3 & 4 & -14 & -14 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -14 \\ -58 \\ 16 \\ -76 \end{bmatrix}$$

Grading: +5 points for the $A^{-1}B$ formula, +5 points for substituting A^{-1}, +5 points for doing the multiplication. Grading for common mistakes: −5 points for AB; +5 points (total) for using another method and getting the correct answer.

5. [15 points] Parameterize the solutions to the system of linear equations whose matrix, in reduced row echelon form, is below. Assume that the original variables were x_1, x_2, x_3, x_4, and x_5.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 2 & -4 \\ 0 & 1 & -1 & 0 & -2 & 8 \\ 0 & 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: The variables which do not have pivots in their columns are x_3 and x_5. They are free variables, so set $x_3 = r$ and $x_5 = s$. The rows of the matrix represent the equations

$$\begin{aligned} x_1 + 2x_5 &= -4 \\ x_2 - x_3 - 2x_5 &= 8 \\ x_4 + 3x_5 &= -4 \end{aligned} \qquad \text{so} \qquad \begin{aligned} x_1 &= -4 - 2x_5 = -4 - 2s \\ x_2 &= 8 + x_3 + 2x_5 = 8 + r + 2s \\ x_4 &= -4 - 3x_5 = -4 - 3s \end{aligned}$$

so the parameterization is

$$\boxed{\begin{aligned} & x_1 = -4 - 2s \\ & x_2 = 8 + r + 2s \\ & x_3 = r \\ & x_4 = -4 - 3s \\ & x_5 = s \\ & \text{where } r, s \text{ can be any real numbers} \end{aligned}}$$

Grading: +5 points for finding the free variables, +5 points for finding the equations for the lead variables, +2 points for writing them together, +3 points for including the "where ... can be any real numbers" condition.

6. [20 points] Find $\begin{vmatrix} 2 & -3 & -2 & 1 \\ 2 & -2 & 3 & -1 \\ 0 & -3 & -1 & -3 \\ 0 & 3 & 0 & 0 \end{vmatrix}$

Solution: Two methods will be presented below. First, expansion by minors, followed by Sarrus's method. (∗ denotes a matrix whose entries are unimportant.)

$$\begin{vmatrix} 2 & -3 & -2 & 1 \\ 2 & -2 & 3 & -1 \\ 0 & -3 & -1 & -3 \\ 0 & 3 & 0 & 0 \end{vmatrix} \underset{EM\,:\,R4}{=\!=\!=\!=} 0 \cdot - |\,*\,| + 3 \cdot + \begin{vmatrix} 2 & -2 & 1 \\ 2 & 3 & -1 \\ 0 & -1 & -3 \end{vmatrix} + 0 \cdot - |\,*\,| + 0 \cdot + |\,*\,|$$

$$= 3[(-18) + (0) + (-2) - (0) - (2) - (12)] = \boxed{-102.}$$

Gaussian elimination:

$$\begin{vmatrix} 2 & -3 & -2 & 1 \\ 2 & -2 & 3 & -1 \\ 0 & -3 & -1 & -3 \\ 0 & 3 & 0 & 0 \end{vmatrix} \underset{②-①}{=\!=\!=\!=} \begin{vmatrix} 2 & -3 & -2 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & -3 & -1 & -3 \\ 0 & 3 & 0 & 0 \end{vmatrix} \underset{\substack{③+3②\\ ④-3②}}{=\!=\!=\!=} \begin{vmatrix} 2 & -3 & -2 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 14 & -9 \\ 0 & 0 & -15 & 6 \end{vmatrix}$$

$$\underset{③+④}{=\!=\!=\!=} \begin{vmatrix} 2 & -3 & -2 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -15 & 6 \end{vmatrix} \underset{④-15③}{=\!=\!=\!=} \begin{vmatrix} 2 & -3 & -2 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 51 \end{vmatrix}$$

$$= 2 \cdot 1 \cdot (-1) \cdot 51 = \boxed{-102.}$$

Grading for common mistakes: -2 *points for not following the* $+-+-$ *pattern;* -10 *points for bad row operations;* -5 *points for not keeping track of the effects of row operations on the determinant;* -5 *points for bad planning.*