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APM 503 Inner Products

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A.1.1

(An inner product is uniquely determined by the norm) Let X be a vector space with inner product $\langle\cdot,\cdot\rangle$ and associated norm $||\cdot||$.

(a) Show that $\langle u,v\rangle + \langle v,u\rangle = \frac{1}{2}(||u+v||^2 - ||u-v||^2)$.

Proof:

$$||u+v||^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle = ||u||^2 + \langle u,v\rangle + \langle v,u\rangle + ||v||^2$$
$$||u-v||^2 = \langle u-v, u-v\rangle = \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle = ||u||^2 - \langle u,v\rangle - \langle v,u\rangle + ||v||^2$$

$$||u+v||^2 - ||u-v||^2 = 2\langle u,v\rangle + 2\langle v,u\rangle \implies \langle u,v\rangle + \langle v,u\rangle = \frac{1}{2}(||u+v||^2 - ||u-v||^2)$$

(b) Show that in a real inner product space $\langle u,v\rangle = \frac{1}{4}(||u+v||^2 - ||u-v||^2)$.

Proof:

$$\begin{aligned} ||u+v||^2 &= \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle = ||u||^2 + \langle u,v\rangle + \langle u,v\rangle + ||v||^2 \\ &= ||u||^2 + 2\langle u,v\rangle + ||v||^2 \\ ||u-v||^2 &= \langle u-v, u-v\rangle = \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle = ||u||^2 - \langle u,v\rangle - \langle u,v\rangle + ||v||^2 \\ &= ||u||^2 - 2\langle u,v\rangle + ||v||^2 \end{aligned}$$

$$||u+v||^2 - ||u-v||^2 = 4\langle u,v\rangle \implies \langle u,v\rangle = \frac{1}{4}(||u+v||^2 - ||u-v||^2)$$

(c) Show that, if X is a complex inner product space,

$$\langle u,v\rangle - \langle v,u\rangle = \frac{i}{2}(||u+iv||^2 - ||u-iv||^2)$$

and

$$\langle u,v\rangle = \frac{1}{4}(||u+v||^2 - ||u-v||^2 + i||u+iv||^2 - i||u-iv||^2).$$

Proof:

(i)

$$||u+iv||^2 = ||u||^2 + \langle u,iv\rangle + \langle iv,u\rangle + ||iv||^2$$
$$||u-iv||^2 = ||u||^2 - \langle u,iv\rangle - \langle iv,u\rangle + ||iv||^2$$

$$||u+iv||^2 - ||u-iv||^2 = 2\langle u,vi\rangle + 2\langle vi,u\rangle = -2i\langle u,v\rangle + 2i\langle v,u\rangle$$
$$\text{Hence } \langle u,v\rangle - \langle v,u\rangle = \frac{i}{2}(||u+iv||^2 - ||u-iv||^2)$$

(ii)

$$||u+v||^2 = ||u||^2 + \langle u,v\rangle + \langle v,u\rangle + ||v||^2$$
$$||u-v||^2 = ||u||^2 - \langle u,v\rangle - \langle v,u\rangle + ||v||^2$$
$$||u+iv||^2 = ||u||^2 + \langle u,iv\rangle + \langle iv,u\rangle + ||iv||^2$$
$$||u-iv||^2 = ||u||^2 - \langle u,iv\rangle - \langle iv,u\rangle + ||iv||^2$$

$$||u+v||^2 - ||u-v||^2 = 2\langle u,v\rangle + 2\langle v,u\rangle$$
$$i||u+iv||^2 - i||u-iv||^2 = 2i\langle u,vi\rangle + 2i\langle vi,u\rangle = 2\langle ui,vi\rangle - 2\langle v,u\rangle = 2\langle u,v\rangle - 2\langle v,u\rangle$$
$$||u+v||^2 - ||u-v||^2 + i||u+iv||^2 - i||u-iv||^2 = 4\langle u,v\rangle$$
$$\text{Hence } \langle u,v\rangle = \frac{1}{4}(||u+v||^2 - ||u-v||^2 + i||u+iv||^2 - i||u-iv||^2)$$

A.1.2

A real $n \times n$ matrix $A = (\alpha_{ij})$ iw called symmetric if $\alpha_{ij} = \alpha_{ji}$ for all $i, j = 1, ..., n$.

(a) Show that a real $n \times n$ matrix A is symmetric if and only if $x \cdot (Ay) = (Ax) \cdot y$ for all $x, y \in \Re^n$.

Here $\cdot$ denotes the Euclidean inner product on $\Re^n$.

Proof:

$(\Longrightarrow)$: Let A be an $n \times n$ matrix that is symmetric. Then $x \cdot (Ay) = \sum_{i=1}^n [x_i(\sum_{j=1}^n \alpha_{ij} y_j)] = \sum_{i=1}^n \sum_{j=1}^n x_i \alpha_{ij} y_j$ so $(Ax) \cdot y = x \cdot (Ay)$.

$(\Longleftarrow)$: Let $x \cdot (Ay) = (Ax) \cdot y$ for $x, y \in \Re^n$, define $v^k \in \Re^n$ with $v^k_{i \neq k} = 0$ and $v^k_{i=k} = 1$. Now let $i, j \in \{1, 2, ..., n\}$ then $v^i \cdot (Ae^j) = \alpha_{ij}$ and $(v^i A) \cdot v^j = \alpha_{ji}$ by assumption $e^i \cdot (Ae^j) = (Ae^i) \cdot e^j$ so $\alpha_{ij} = \alpha_{ji}$ so A is symmetric.

A symmetric matrix A is called positive definite if $x \cdot (Ax) > 0$ for all $x \in \Re^n$, $x \neq 0$.

(b) Show: A function $\langle , \rangle$ from $\Re^n \times \Re^n$ into $\Re$ is an inner product on $\Re^n$ if and only if there exists a positive definite symmetric matrix A such that $\langle x, y \rangle = x \cdot (Ay)$ for all $x, y \in \Re^n$.

Proof:

$(\Longrightarrow)$: Suppose $\langle , \rangle$ from $\Re^n \times \Re^n$ to $\Re$ is an inner product space. Define $v^k \in \Re^n$ with $v^k_{i \neq k} = 0$ and $v^k_{i=k} = 1$. Let $A = (\alpha_{ij})$ be a real $n \times n$ matrix with $\alpha_{ij} = \langle v^i, v^j \rangle$. Since $\mathbf{K} = \Re$, $\alpha_{ij} = \langle v^i, v^j \rangle = \langle v^j, v^i \rangle = \alpha_{ji}$, so A is symmetric. Let $x, y \in \Re^n$ notice that $x = \sum_{i=1}^n x_i v^i$ and $y = \sum_{i=1}^n y_i v^i$. So $\langle x, y \rangle = \langle \sum_{i=1}^n x_i v^i, y \rangle = \sum_{i=1}^n x_i \langle v^i, y \rangle = \sum_{i=1}^n x_i \langle v^i, \sum_{j=1}^n y_j v^j \rangle = \sum_{i=1}^n \sum_{j=1}^n x_i y_j \langle v^i, v^j \rangle = \sum_{i=1}^n \sum_{j=1}^n x_i y_j \alpha_{ij} = x \cdot (Ay)$. Now $x \cdot (Ax) = \langle x, x \rangle > 0$ for all $x \in \Re^n$, $x \neq 0$, so A is positive definite.

$(\Longleftarrow)$: Let A be a positive definite symmetric $n \times n$ matrix with $\langle x, y \rangle = x \cdot (Ay)$ for all $x, y \in \Re^n$. Then for $u, v \in \Re^n$, $\alpha \in \mathbf{K}$, $\langle u, v \rangle = u \cdot (Av) = (Au) \cdot v = \overline{v \cdot (Au)} = \overline{\langle v, u \rangle}$

$$\langle \alpha u, v \rangle = \alpha u \cdot (Av) = \alpha(u \cdot (Av)) = \alpha \langle u, v \rangle$$

$$\langle u + v, w \rangle = (u + v) \cdot (Aw) = (u \cdot Aw) + (v \cdot Aw) = \langle u, w \rangle + \langle v, w \rangle$$

$\langle u, u \rangle = u \cdot (Au) > 0$ for $u \neq 0$ by positive definite.

A.1.3

Let A be a positive definite symmetric $n \times n$ matrix and $\cdot$ denote the inner product on $\Re^n$. Show: $|x \cdot (Ay)|^2 \leq [x \cdot (Ax)][y \cdot (Ay)]$ for all $x, y \in \Re^n$ with equality holding if and only if x and y are linearly dependent.

Proof:

Recall by Exercise $A.1.2$ $b)$ $x \cdot (Ay) = \langle x, y \rangle$

Now by Theorem $A.2$ and $|x \cdot (Ay)|^2 = |\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle = [x \cdot (Ax)][y \cdot (Ay)]$ and also by Theorem $A.2$ the equality holds for iff x, y are linearly dependent.

A.1.4

Consider $\ell^2 = \{x = (x_n) \in \mathbf{C}^{\mathbf{N}}; ||x||_2^2 < \infty\}$ where

$$||x||_2^2 = \sum_{n=1}^{\infty} |x_n|^2.$$

Show:

(a) For each $x = (x_n)$ and $y = (y_n)$ in ℓ^2, the series

$$\sum_{k=1}^{\infty} x_k \overline{y_k} =: \langle x, y \rangle$$

converges in $\mathbf{C}$ (with absolute value) and defines an inner product on ℓ^2.

Proof:

Note that $y = (y_n) \in \ell^2 \implies z = (z_k) = (\overline{y_n}) \in \ell^2$. Note $|y|, |z| \in \Re^{\mathbf{N}}$ so for all $m \in \mathbf{N}$,

$$|\sum_{k=1}^{m} x_k \overline{y_k}| \leq \sum_{k=1}^{m} |x_k z_k| = \sum_{k=1}^{m} |x_k||z_k| \leq (\sum_{k=1}^{m} |x_k|^2)^{1/2} \cdot (\sum_{k=1}^{m} |z_k|^2)^{1/2} \leq (\sum_{k=1}^{\infty} |x_k|^2)^{1/2} \cdot (\sum_{k=1}^{\infty} |z_k|^2)^{1/2}$$

$$= (||x_k||_2^2)^{1/2} \cdot (||z_k||_2^2)^{1/2} < \infty \text{ by Cauchy-Schwarz in } \Re^n$$

so the series converges. Now, let $x, y, z \in \ell^2$, $\alpha \in \mathbf{C}$. Then

(i)

$$\overline{\langle x, y \rangle} = \overline{\sum_{k=1}^{\infty} x_k \overline{y_k}} = \overline{\lim_{n\to\infty} \sum_{k=1}^{n} x_k \overline{y_k}} = \overline{\lim_{n\to\infty} \sum_{k=1}^{n} Re(x_k \overline{y_k}) + i \cdot Img(x_k \overline{y_k})}$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} Re(x_k \overline{y_k}) - i \cdot Img(x_k \overline{y_k}) = \lim_{n\to\infty} \sum_{k=1}^{n} \overline{x_k \overline{y_k}}$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} y_k \overline{x_k} = \sum_{k=1}^{\infty} y_k \overline{x_k} = \langle y, x \rangle$$

(ii)

$$\langle \alpha x, y \rangle = \sum_{k=1}^{\infty} \alpha x_k \overline{y_k} = \alpha \sum_{k=1}^{\infty} x_k \overline{y_k} = \alpha \langle x, y \rangle$$

(iii)

$$\langle x + y, z \rangle = \sum_{k=1}^{\infty} (x_k + y_k) \overline{z_k} = \sum_{k=1}^{\infty} x_k \overline{z_k} + \sum_{k=1}^{\infty} y_k \overline{z_k} = \langle x, z \rangle + \langle y, z \rangle$$

(iv)

$$\langle x, x \rangle = \sum_{k=1}^{\infty} x_k \overline{x_k} = \sum_{k=1}^{\infty} |x_k|^2 > 0 \text{ for } x \neq \mathbf{O}$$

(b) ℓ^2 with this product is a Hilbert space.

Hint: Modify the proof of Theorem 2.26.

Proof:

Let $(x^n)_{n\in\mathbf{N}} \subset \ell^2$ be a Cauchy sequence, let $x^n = (x_j^n)_{j\in\mathbf{N}} = (x_1^n, x_2^n, ...) \in \ell^2$, then for $x = (x_j)_{j\in\mathbf{N}}$, $y = (y_j)_{j\in\mathbf{N}} \in \ell^2$, $||x - y||_2 = (\sum_{j=1}^{\infty} |x_j - y_j|^2)^{1/2}$. Now consider $\epsilon > 0$, then there exits $N \in \mathbf{N}$, such that if $m, n > N$ then $||x^m - x^n||_2 < \epsilon$. Thus for all $j \in \mathbf{N}$,

$$|x_j^m - x_j^n|^2 \leq \sum_{j=1}^{\infty} |x_j^m - x_j^n|^2 = ||x^m - x^n||_2^2 < \epsilon^2$$

Then since the sequence $(x_j^n)_{j\in\mathbf{N}} \subset \mathbf{C}$ is Cauchy, and $\mathbf{C}$ is complete, for all $j \in \mathbf{N}$ there exists a $x_j \in \mathbf{C}$ such that $lim_{n\to\infty} x_j^n = x_j$. Now consider an arbitrary but fixed $k \in \mathbf{N}$, then if $m, n > N$

$$\sum_{j=1}^{k} |x_k^m - x_j^n|^2 \leq \sum_{j=1}^{\infty} |x_j^m - x_j^n|^2 = ||x^m - x^n||_2^2 < \epsilon^2 \qquad (1)$$

Hence for $n \to \infty$ and $n > M$ $\quad \sum_{j=1}^{k} |x_j^m - x_j|^2 < \epsilon^2$

$$(\sum_{j=1}^{k} |x_j|^2)^{1/2} \leq (\sum_{j=1}^{k} |x_j^m - x_j|^2)^{1/2} + (\sum_{j=1}^{k} |x_j^m|^2)^{1/2} < \epsilon + (\sum_{j=1}^{k} |x_j^m|), \text{ (Property of Euclidean norm)}$$

Then for $k \to \infty$, $||x||_2 \leq \epsilon + ||x^m||_2$ hence $x = (x_j)_{j\in\mathbf{N}} \in \ell^2$ also, for $k \to \infty$ and $m > N$, $||x^m - x||_2^2 = \sum_{j=1}^{\infty} |x_j^m - x_j|^2 < \epsilon^2$ implying $lim_{m\to\infty} ||x^m - x||_2 = 0$. Therefore $(x^m)_{m=1}^{\infty} \subset l^p$, is a convergent sequence that converges to $x \in \ell^2$. We conclude then that l^2 is a complete metric space and an inner product space hence a Hilbert space.

A.1.5

Let X be an inner product space over $\mathbf{K}$ and $(x_n), (y_n)$ be Cauchy sequences in X. Show: The sequence $(\langle x_n, y_n \rangle)$ converges in $\mathbf{K}$.

Proof: Since $(x_n), (y_n)$ are Cauchy sequences over a metric space they are bounded, so there exists $M \in \mathbf{K}$ such that $||x_n||, ||y_n|| < M$ for $n \in \mathbf{N}$. Since $(x_n), (y_n)$ are Cauchy, there exists $N \in \mathbf{N}$ such that $||x_m - x_n|| < \frac{\epsilon}{2}$ and $||y_m - y_n|| < \frac{\epsilon}{2}$ for $n > N$. Then, using the Cauchy-Schwarz inequality

$$\begin{aligned} |\langle x_n, y_n \rangle - \langle x_m, y_m \rangle| &= |\langle x_n, y_n \rangle - \langle x_m, y_n \rangle + \langle x_m, y_n \rangle - \langle x_m, y_m \rangle| \\ &= |\langle x_n - x_m, y_n \rangle + \langle x_m, y_n - y_m \rangle| \\ &\leq ||x_n - x_m|| \cdot ||y_n|| + ||x_m|| \cdot ||y_n - y_m|| \\ &\leq ||x_n - x_m|| M + M ||y_n - y_m|| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

So $(\langle x_n, y_n \rangle)$ is a Cauchy sequence in $\mathbf{K}$ and therefore a convergent sequence in $\mathbf{K}$.

A.1.6

Let X be an inner product space and x, y be points in X, $\alpha \in \mathbf{K}$, and $(x_n), (y_n)$ be sequences in X and (α_n) a sequence in $\mathbf{K}$.

Show: If $x_n \to x$, $y_n \to y$ and $\alpha_n \to \alpha$ as $n \to \infty$, then $\langle \alpha_n x_n, y_n \rangle \to \langle \alpha x, y \rangle$ as $n \to \infty$.

Proof:

Let $\epsilon > 0$, then since (y_n) converges, there is $N \in \mathbf{N}$ such that $||y_n - y|| < \min\{\frac{\epsilon}{2}, \frac{\epsilon}{2||x||+1}\}$ for $n \geq N$, notice that $||y_n|| < ||y|| + \epsilon$ for $n \geq N$. Now, since (x_n) converges, there is $M \in \mathbf{N}$ with $M > N$ and $||x_n - x|| < \frac{\epsilon}{2(||y||+\epsilon)}$. Consider the sequence $\langle x_n, y_n \rangle$

$$\begin{aligned} |\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n - x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle| \\ &= |\langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle| \\ &\leq |\langle x_n - x, y_n \rangle| + |\langle x, y_n - y \rangle| \\ &\leq ||x_n - x|| \cdot ||y_n|| + ||x|| \cdot ||y_n - y|| \\ &\leq ||x_n - x|| \cdot ||y + \epsilon|| + ||x|| \cdot ||y_n - y|| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for } n \geq M \end{aligned}$$

So $\langle x_n, y_n \rangle \to \langle x, y \rangle$. Then since $\alpha_n \to \alpha$, $\langle \alpha_n x_n, y_n \rangle = \alpha_n \langle x_n, y_n \rangle \to \alpha \langle x, y \rangle = \langle \alpha x, y \rangle$, as $n \to \infty$.

A.1.7

Let X be an inner product space, $x \in X$ and (x_n) a sequence in X. Show: $x_n \to x$ as $n \to \infty$ if and only if $||x_n|| \to ||x||$ and $\langle x_n, x \rangle \to \langle x, x \rangle$ as $n \to \infty$.

Proof:

$(\Longrightarrow)$: Consider the sequence (x_n) in X let $x \in X$ such that $x_n \to x$ then $0 \leq |||x_n|| - ||x||| \leq ||x_n - x|| \to 0$ hence $||x_n|| \to ||x||$ now let $\epsilon > 0$ with $||x_n - x|| < \frac{\epsilon}{||x||}$, $0 \leq |\langle x_n, x \rangle - \langle x, x \rangle| = |\langle x_n - x, x \rangle| \leq ||x_n - x|| ||x|| < \frac{\epsilon}{||x||} ||x|| = \epsilon$. Hence $\langle x_n, x \rangle \to \langle x, x \rangle$.

$(\Longleftarrow)$: Consider the sequence (x_n) in X let $x \in X$ such that $||x_n|| \to ||x||$ as $n \to \infty$ and $\langle x_n, x \rangle \to \langle x, x \rangle$, then $||x_n - x||^2 = \langle x_n - x, x_n - x \rangle = \langle x_n, x_n \rangle - \langle x_n, x \rangle - \langle x, x_n \rangle + \langle x, x \rangle = ||x_n||^2 + ||x||^2 - \langle x_n, x \rangle - \langle x, x_n \rangle \to ||x||^2 + ||x||^2 - \langle x, x \rangle - \langle x, x \rangle = 0$
Hence $x_n \to x$ as $n \to \infty$.

A.1.8

Let X be an inner produce space. Let $y \in X$ be fixed but arbitrary. Define $f, g : X \to \mathbf{C}$ by

$$f(x) = \langle x, y \rangle, \qquad g(x) = \langle y, x \rangle, \qquad x \in X$$

Then f and g are Lipschitz continuous with Lipschitz constant $||y||$.

Proof:

Let $x, y, z \in X$,

$$d(f(x), f(z)) = |\langle x, y\rangle - \langle z, y\rangle| = |\langle x - z, y\rangle| \leq ||x - z|| \cdot ||y|| = ||y|| d(x, z)$$

$$d(g(x), g(z)) = |\langle y, x\rangle - \langle y, z\rangle| = |\langle y, x - z\rangle| \leq ||y|| \cdot ||x - z|| = ||y|| d(x, z)$$

Notice $||\langle x - z, y\rangle|| \leq ||x - z|| \cdot ||y||$ is true by the Cauchy-Schwarz inequality. Hence f, g are Lipschitz continuous and $||y||$ is an upper bound on the Lipschitz constant. Now, observe that this upper bound is achieved: For $y \neq 0$,

$$\frac{d(f(y), f(0))}{d(y, 0)} = \frac{|\langle y, y\rangle - \langle 0, y\rangle|}{||y||} = \frac{|\langle y, y\rangle|}{||y||} = \frac{||y||^2}{||y||} = ||y||$$

$$\frac{d(g(y), g(0))}{d(y, 0)} = \frac{|\langle y, y\rangle - \langle y, 0\rangle|}{||y||} = \frac{|\langle y, y\rangle|}{||y||} = \frac{||y||^2}{||y||} = ||y||$$

If $y = 0$ then $d(y, 0) = d(0, 0) = 0 = ||y|| \cdot 0 = ||y|| d(f(0), f(0)) = ||y|| d(f(y), f(0))$.

So the Lipschitz constant for f and g is $||y||$.

1 A.1.9

Let M be a complete linear subspace of the inner product space X.

Show: Each vector $u \in X$ has a unique representation $u = v + w$ such that $v \in M$ and $\langle w, z\rangle = 0$ for all $z \in M$.

Remark: The vector $v \in M$ is called the *orthogonal projection* of u on M and is also characterized as the unique vector in M such that $d(u, M) = ||u - v||$ which exists according to Proposition $A.9$

Hint: set $w = u - v$. Let $z \in M$. Observe that the function $\phi(\alpha) = ||w - \alpha z||^2$, $\alpha \in \mathbf{K}$, has a minimum at $\alpha = 0$.

Proof: Let $u \in X$. Since M is a linear subspace of X it is convex, so there is a unique $v \in M$ such that $d(u, M) = ||u - v||$. Set $w = u - v$ and let $z \in M$. Consider the function $\phi(\alpha) = ||w - \alpha z||^2$, $\alpha \in \mathbf{K}$. Notice that $\phi(\alpha) = ||u - (v + \alpha z)||^2$ and that $(v + \alpha z) \in M$ so $\phi(\alpha)$ is minimized when $\alpha = 0$, since this is an extremum, $\phi_\alpha(0) = 0$. Consider $\alpha \in \mathbf{K}$ then,

$$\begin{aligned}
\phi(\alpha) &= ||w - \alpha z||^2 \\
&= \langle w - \alpha z, w - \alpha z\rangle \\
&= ||w||^2 - \bar{\alpha}\langle w, z\rangle - \alpha\langle z, w\rangle + |\alpha|^2 ||z||^2 \\
&= ||w||^2 + |\alpha|^2 ||z||^2 - \alpha\left(\langle w, z\rangle + \langle z, w\rangle\right) \\
\implies \phi_\alpha(\alpha) &= 2|\alpha| ||z||^2 - \left(\langle w, z\rangle + \langle z, w\rangle\right) \\
\implies \phi_\alpha(0) &= -\left(\langle w, z\rangle + \langle z, w\rangle\right) = 0 \implies \Re(\langle w, z\rangle) = 0
\end{aligned}$$

Similarly we have for $\alpha \in \Re$,

$$\begin{aligned}
\phi(\alpha i) &= ||w - \alpha z i||^2 \\
&= \langle w - \alpha z i, w - \alpha z i\rangle \\
&= ||w||^2 + \alpha i\langle w, z\rangle - \alpha i\langle z, w\rangle + |\alpha|^2 ||z||^2 \\
&= ||w||^2 + |\alpha|^2 ||z||^2 + \alpha i\left(\langle w, z\rangle - \langle z, w\rangle\right) \\
&= ||w||^2 + |\alpha|^2 ||z||^2 - 2\alpha\Im(\langle w, z\rangle) \\
\implies \phi_\alpha(\alpha i) &= 2|\alpha| ||z||^2 - 2\Im(\langle w, z\rangle) \\
\implies \phi_\alpha(0i) &= -2\Im(\langle w, z\rangle) = 0 \implies \Im(\langle w, z\rangle) = 0
\end{aligned}$$

Therefore we conclude that $\langle w, z\rangle = \Re(\langle w, z\rangle) + i\Im(\langle w, z\rangle) = 0 + 0i = 0$.

Uniqueness: Let $v, x \in M$ then consider $u = v+w$ and $u = x+y$ therefore $\langle w, z\rangle = 0 = \langle y, z\rangle$ for all $z \in M$ then consider $z \neq \mathbf{0}$ then if $\langle w, z\rangle = 0 = \langle y, z\rangle$ hence $\langle w, z\rangle - \langle y, z\rangle = \langle w - y, z\rangle = 0$ therefore by Cauchy-Schwarz, $\langle w - y, z\rangle = ||w - y|| \cdot ||z|| = 0$ so $|w - y| = 0$ since $z \neq \mathbf{0}$ which implies $w = y$. Thus $v = x$ because $u = v + w = x + y \implies v + w = x + w$ since $y = w$. So $v - x = y - w$. Notice that $v - x \in M$ and $\langle y - w, z\rangle = 0$ for all $z \in M$. In particular $0 = \langle y - w, v - x\rangle = \langle y - w, y - w\rangle$. By the properties of the inner product, $0 = y - w = v - x$.