Final Exam MAT 272 - Gelb

December 9, 2014

Name KEY

Student ID No. ____________

Directions:

- There are 4 multiple choice questions worth 8 points each, 5 true false questions worth 4 points each, and 2 free response questions worth 24 points each.
- You must show your work on all questions, **including multiple choice.**
- Fort true/false questions, you must give a clear and correct explanation to justify your answer. If the anser is false, you may use a counter example.
- Partial credit is only available on the free response problems.
- Read all the questions carefully.
- You may not use your calculators to graph functions or integrate integrals.
- Put the final answer to the space provided in an orderly fashion. Box your final answers. No partial credit will be given if more than one answer is given, or if it unclear which answer is meant to be your final answer.

Honor Statement

By signing below you confirm that you have neither given nor received any unauthorized assistance on this exam. This includes any use of a graphing calculator beyond those uses specifically authorized by the School of Mathematical and Statistical Sciences and your instructor. Furthermore, you agree not to discuss this exam with anyone in any section of MAT 272 until the exam testing period is over. In addition, your calculator's program memory and menus may be checked at any time and cleared by any exam proctor.

Signature:

Multiple Choice

1. Which of the following are conservative vector fields?

   ✓ (a) $< e^x \cos y + yz, xz - e^x \sin y, xy + z >$

   ✓ (b) $< yz, xz, xy >$

   ✗ (c) $< xz, yz, xy >$

   (d) (a) and (b)

   (e) (a) and (c)

   (f) (b) and (c)

   (g) (a), (b), and (c)

To get $\vec{F} = \nabla \varphi \iff \langle f, g, h \rangle = \langle \varphi_x, \varphi_y, \varphi_z \rangle$

We need $f_y = g_x$; $f_z = h_x$; $g_z = h_y$

a) $f_y = -e^x \sin y + 1$, $g_x = 1 - e^x \sin y$ ✓

$f_z = y$; $h_x = y$ ✓, $g_z = x$, $h_y = x$ ✓

b) $f_y = z$, $g_x = z$ ✓, $f_z = y$, $h_x = y$ ✓, $g_z = x$, $h_y = x$ ✓

c) $f_y = 0$, $g_x = 0$ ✓, $f_z = x$, $h_x = y$. ($f_z \neq h_x$)

2. If a parametric surface given by $\mathbf{r}_1(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$, with $-3 \leq u \leq 3, -5 \leq v \leq 5$ has surface area equal to 4, the surface area of the parametric surface given by $\mathbf{r}_2(u,v) = 4\mathbf{r}_1(u,v)$, with $-3 \leq u \leq 3, -5 \leq v \leq 5$ is

   (a) 16

   (b) $\frac{1}{4}$

   (c) 8

   (d) 64

   (e) Not enough information or none of the above.

surface area for parametric surface given by $\vec{r}_1$ $\displaystyle\int_{u=-3}^{3} \int_{v=-5}^{5} |\vec{t}_u \times \vec{t}_v| \, dv\, du = 4$

for $\vec{r}_1$: $\vec{t}_{1,u} \times \vec{t}_{1,v} = \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \langle y_u z_v - z_u y_v, x_v z_u - x_u z_v, x_u y_v - y_u x_v \rangle$

$\vec{r}_1(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

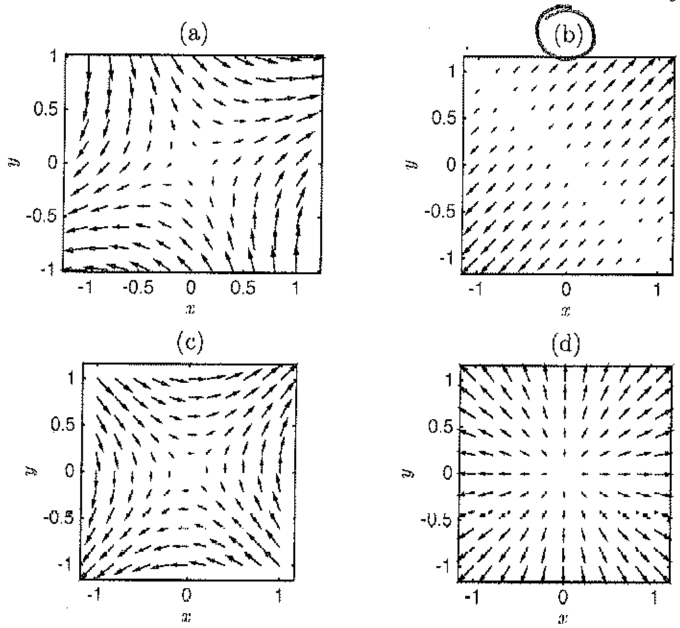
for $\vec{r}_2 = 4\vec{r}_1 = \langle 4x, 4y, 4z \rangle$: $\vec{t}_u \times \vec{t}_v = \begin{vmatrix} i & j & k \\ 4x_u & 4y_u & 4z_u \\ 4x_v & 4y_v & 4z_v \end{vmatrix} = \langle 16(y_u z_v - z_u y_v), 16(x_v z_u - x_u z_v), 16(x_u y_v - y_u x_v) \rangle$

$= 16(\vec{t}_{1,u} \times \vec{t}_{1,v})$

$\Rightarrow$ surface area for parametric surface given by $\vec{r}_2$ is

$16 \cdot 4 = 64$

3. Which of the following is a graphical representation of $\mathbf{F} = < x+y, x+y >$



$\langle 0,0 \rangle$ when $y = -x$

$\hat{i}, \hat{j}$ components positive for $x+y > 0$

$\hat{i}, \hat{j}$ components negative for $x+y < 0$

(e) Not enough information or none of the above

4. The work done by the force field $\mathbf{F} = < 3x, 3y, 3 >$ on a particle that moves along the helix $\mathbf{r}(t) = < 5\cos t, 5\sin t, 7t >$ from $(5,0,0)$ to $(5,0,14\pi)$ is

(a) 150π

(b) $150\pi + 42\pi^2$

(c) 42π

(d) 6π

(e) Not enough information or none of the above

$\int_C \vec{F} \cdot d\vec{r}$

$\vec{r}(t) = \langle 5\cos t, 5\sin t, 7t \rangle$

is at $(5,0,0)$ when $t = 0$

is at $(5,0,14\pi)$ when $t = 2\pi$

$\int_0^{2\pi} \langle 3(5\cos t), 3(5\sin t), 3 \rangle \cdot \langle -5\sin t, 5\cos t, 7 \rangle dt$

$= \int_0^{2\pi} (-75\cos t\sin t + 75\cos t\sin t + 21)\, dt = \int_0^{2\pi} 21\, dt = 42\pi$

5. Assume $\mathbf{F}$ is a sufficiently differentiable vector field and ϕ is a sufficiently differentiable scalar-valued function. Consider the following expressions:

- X (i) $\nabla(\nabla \cdot \phi)$ $\quad$ $\nabla\varphi$ is a scalar
- (ii) $\nabla \times \nabla\phi$
- X (iii) $\nabla \times (\nabla \cdot \mathbf{F})$ $\quad$ $\nabla \cdot F$ scalar
- X (iv) $\nabla\mathbf{F}$ $\quad$ $\vec{F}$ is a vector
- (v) $\nabla \times (\nabla \times \mathbf{F})$

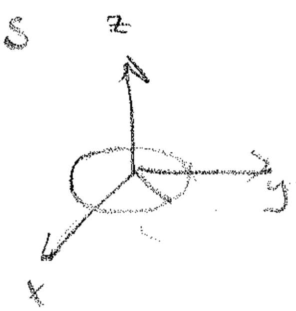
Which of the above expressions make sense?

(a) (i) and (v)
(b) (ii) and (v)
(c) (ii), (iv), and (v)
(d) (i), (iii), and (v)
(e) all of the above

6. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle xz, yz, xy \rangle$, and C is the circle $x^2 + y^2 = 4$ in the xy plane with counterclockwise orientation. (Hint: Use Stoke's theorem)

(a) π
(b) 0
(c) 1
(d) 2π
(e) Not enough information or none of the above

Stokes thm $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$



$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & yz & xy \end{vmatrix} = \langle x - z,\ x - y,\ 0 \rangle$$

$\vec{n} = \langle 0, 0, 1 \rangle$ since circle is in xy plane

$\Rightarrow \quad (\nabla \times \vec{F}) \cdot \vec{n} = 0$

True/False: Make sure to **justify** your answer!

1. The vector field $\mathbf{F} = < f(x), g(y) >$ is conservative on $\mathbb{R}^2$.

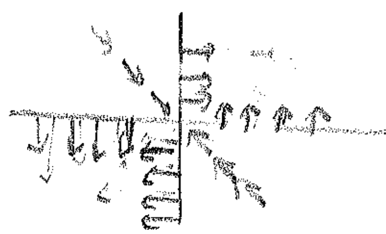
True $\quad f_y = 0 = g_x$

2. The vector field $\mathbf{F} = < y, x >$ has both zero circulation along and zero flux across the unit circle centered at the origin.

True $\quad f_y = 1 = g_x$ and $f_x + g_y = 0$

3. The vector field $\mathbf{F} = \frac{<y,x>}{\sqrt{x^2+y^2}}$ is a rotation field.

False



4. Two vector fields in $\mathbb{R}^3$ with the same divergence differ by a constant vector field.

If $\nabla \cdot F_1 = \nabla \cdot F_2$

then $f_{1x} + g_{1y} + h_{1z} = f_{2x} + g_{2y} + h_{2z}$

False for example $f_1 = f(x) + g(y,z)$ and $f_2 = f(x) + g(y,z)$

5. If $\mathbf{F} = < x, y, z >$ and S encloseds a region D, then $\int\int_S \mathbf{F} \cdot \mathbf{n} dS$ is three times the volume of D.

True by divergence thm $\iint_S \vec{F} \cdot \vec{n}\, dS$

$= \iiint_D \nabla \cdot F\, dV = \iiint_D (1+1+1)\, dV = 3 \iiint_D dV$

Free Response

1. Let $\mathbf{F} =< z^2x, \frac{1}{3}y^3 + \tan z, x^2z + y^2 >$ and S be the top half of the sphere $x^2 + y^2 + z^2 = 1$. Note that S is not a closed surface.

   (a) (8 points) Let S_1 be the disk $x^2 + y^2 \le 1$ oriented downward, and define $S_2 = S \cup S_1$. Calculate $\int\int_{S_2} \mathbf{F} \cdot \mathbf{n} dS$. Hint: Use the Divergence Theorem.

   (b) (8 points) Now calculate $\int\int_{S_1} \mathbf{F} \cdot \mathbf{n} dS$.

   (c) (4 points) Use parts (a) and (b) to evaluate $\int\int_S \mathbf{F} \cdot \mathbf{n} dS$.

a) over S_2 we have $\iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

$\nabla\cdot\vec{F} = z^2 + y^2 + x^2 \Rightarrow \nabla\cdot\vec{F} = \rho^2$ in spherical coordinates

S_2 in spherical coordinates is described as

$\{(\rho,\varphi,\theta) : \rho \le 1,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi\}$

$\Rightarrow \iiint_V \nabla\cdot\vec{F}\,dV = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi/2}\int_{\rho=0}^{1} \rho^2(\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta$

$= 2\pi\int_{\varphi=0}^{\pi/2}\left(\int_{\rho=0}^{1}\rho^4 d\rho\right)\sin\varphi\,d\varphi = 2\pi\int_{\varphi=0}^{\pi/2}\left(\frac{\rho^5}{5}\Big|_0^1\right)\sin\varphi\,d\varphi$

$= \frac{2\pi}{5}(-\cos\varphi)\Big|_0^{\pi/2} = \frac{2\pi}{5}$

b) S_1 is the surface enclosed by $x^2+y^2 \le 1$ at $z=0$

parameterizing S_1: $\vec{r}(u,v) = \langle v\cos u, v\sin u, 0\rangle$

(use $u=\theta$, $v=r$)

$\vec{t}_u = \langle -v\sin u, v\cos u, 0\rangle$; $\vec{t}_v = \langle \cos u, \sin u, 0\rangle$

$\Rightarrow \vec{n} = \vec{t}_u \times \vec{t}_v = \langle 0, 0, -v\rangle$ (right direction since pointed downward)

$\Rightarrow \iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \int_{u=0}^{2\pi}\int_{v=0}^{1}(v^2\sin^2 u)(-v)\,dv\,du = \int_{u=0}^{2\pi}\left[\int_{v=0}^{1} -v^3 dv\right]\sin^2 u\,du$

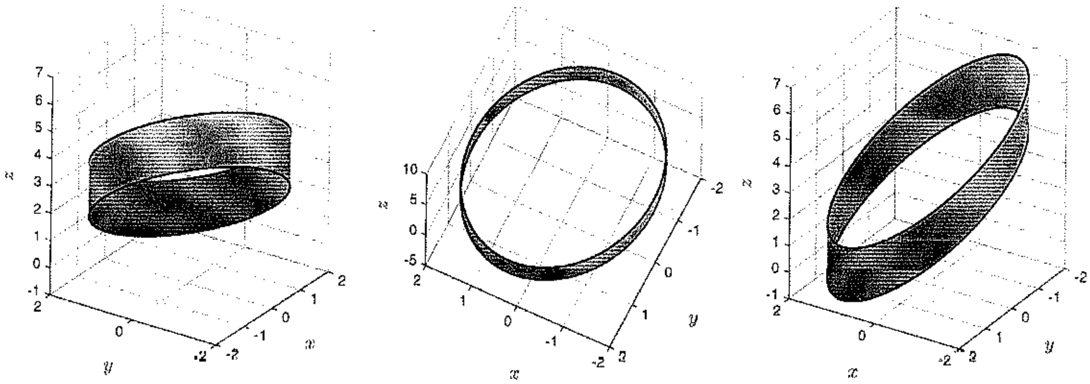
$(x^2z + y^2 = y^2 = v^2\sin^2 u$ $(z=0$ on disk$))$

$= -\frac{1}{4}\int_{u=0}^{2\pi}\sin^2 u\,du = -\frac{1}{4}\left(\frac{u}{2} - \frac{\sin 2u}{4}\right)\Big|_0^{2\pi} = -\frac{\pi}{4}$

c) $\iint_S \vec{F}\cdot\vec{n}\,dS = -\iint_{S_1} F\cdot\vec{n}\,dS + \iiint_V \nabla\cdot\vec{F}\,dV =$

$-\left(-\frac{\pi}{4}\right) + \frac{2\pi}{5} = \frac{5\pi}{20} + \frac{8\pi}{20} = \frac{13}{20}\pi$

3. (10 points) Find the flux of $\mathbf{F} = \langle x, y, z \rangle$ through the curved surface of the cylinder $x^2 + y^2 = 4$ bounded below by the plane $x + y + z = 2$ and above by the plane $x + y + z = 4$ and oriented away from the z-axis. The surface is displayed below from three perspectives (angles of view).



Cylinder: $x^2+y^2=4$

using cylindrical coordinates: $\vec{r}(\theta, r, z) = \langle 2\cos\theta, 2\sin\theta, z \rangle$

parameterize region

R: $\{\vec{r}(u,v) = \langle 2\cos u, 2\sin u, v \rangle ; 0 \le u \le 2\pi, \ 2-(2\cos u + 2\sin u) \le v \le 4-(2\cos u + 2\sin u)\}$

$\vec{t}_u = \langle -2\sin u, 2\cos u, 0 \rangle$; $\vec{t}_v = \langle 0, 0, 1 \rangle$

$\Rightarrow \vec{t}_u \times \vec{t}_v = \langle 2\cos u, 2\sin u, 0 \rangle = \vec{n}$

$\vec{F} = \langle 2\cos u, 2\sin u, v \rangle$ in (u,v) coordinates

$$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\, ds = \int_{u=0}^{2\pi} \int_{v=2-(2\cos u+2\sin u)}^{4-(2\cos u + 2\sin u)} (4\cos^2 u + 4\sin^2 u)\, dv\, du$$

$$= \int_0^{2\pi} 4[4-(2\cos u + 2\sin u) - (2-(2\cos u + 2\sin u))]\, du = \int_0^{2\pi} 8\, du = 16\pi$$

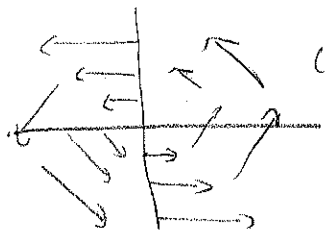
2. Consider the vector field $\mathbf{F} = < x-y, x >$ in $\mathbb{R}^2$.

(a) (6 points) Make a sketch of $\mathbf{F}$.

(b) (2 points) Is $\mathbf{F}$ a conservative vector field?

(c) (6 points) Calculate the circulation around the unit circle with counterclockwise orientation.

(d) (6 points) Calculate the flux of $\mathbf{F}$ across the unit circle with counterclockwise orientation in the xy plane.

a)  (rotational field)

b) If $\vec{F}$ is conservative then $\vec{F} = \nabla\varphi \Rightarrow f_y = g_x$

Since $f_y = -1$ and $g_x = 1$, $\vec{F}$ is not conservative

c) $\oint_C \vec{F}\cdot d\vec{r}$; $\vec{r} = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$ (unit circle) C

$\vec{F} = \langle x-y, x\rangle = \langle \cos t - \sin t, \cos t \rangle$

$\vec{r}\,' = \langle -\sin t, \cos t \rangle$

$\Rightarrow \vec{F}\cdot\vec{r}\,' = -\cos t\sin t + \sin^2 t + \cos^2 t = -\cos t \sin t + 1$

$\int_0^{2\pi} \vec{F}\cdot\vec{r}\,'\,dt = \int_0^{2\pi} (1 - \cos t \sin t)\,dt = 2\pi + \frac{\cos^2 t}{2}\Big|_0^{2\pi} = 2\pi$

(observe: Green's thm $\int_{\theta=0}^{2\pi}\int_{r=0}^{1} (g_x - f_y)\, r\,dr\,d\theta = 2\pi\int_{r=0}^{1} 2r\,dr = 2\pi$)

d) Flux: $\int_C \vec{F}\cdot\vec{n}\,ds$ recall $T = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\sin t, \cos t \rangle$

and $\vec{n} \pm T' = \langle -\cos t, -\sin t \rangle$ (unit length)

to make consistent with counterclockwise orientation; $\vec{n} = \langle \cos t, \sin t \rangle$

$\Rightarrow \int_0^{2\pi} \langle \cos t - \sin t, \cos t \rangle \cdot \langle \cos t, \sin t \rangle\,dt = \int_0^{2\pi} (\cos^2 t - \cos t \sin t + \cos t \sin t)\,dt$

$= \int_0^{2\pi} \cos^2 t\,dt = \frac{t}{2}\Big|_0^{2\pi} + \frac{\sin 2t}{4}\Big|_0^{2\pi} = \pi$

(observe: Green's theorem $\int_{\theta=0}^{2\pi}\int_{r=0}^{1} (\underbrace{f_x + g_y}_{1})\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\,dr\,d\theta = \pi$