TRACES AND EXTENSIONS OF MATRIX-WEIGHTED BESOV SPACES

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Abstract. Let V be a matrix weight on $\mathbb{R}^{n+1}$ and W a matrix weight on $\mathbb{R}^n$, satisfying, for example, the matrix A_p condition. Define the trace, or restriction, operator Tr by $Tr(f)(x') = f(x', 0)$, where $x' \in \mathbb{R}^n$ and f is a function on $\mathbb{R}^{n+1}$. If $\alpha > \frac{1}{p} + n\left(\frac{1}{p} - 1\right)_+ + \frac{\beta - n}{p}$, where β is the doubling exponent of W, then the trace operator is bounded from $\dot{B}_p^{\alpha q}(V)$ into $\dot{B}_p^{\alpha - \frac{1}{p}, q}(W)$ (matrix-weighted Besov spaces), if and only if the weights V and W satisfy

$$\frac{1}{|I|}\int_I \|W^{1/p}(t)\vec{y}\|^p\,dt \le C\frac{1}{|Q(I)|}\int_{Q(I)} \|V^{1/p}(t)\vec{y}\|^p\,dt$$

for all $\vec{y}$ and all dyadic cubes $I \subseteq \mathbb{R}^n$, where $Q(I) = I \times [0, \ell(I)]$. If V and W satisfy the converse inequality, then there exists a continuous linear map $Ext : \dot{B}_p^{\alpha - \frac{1}{p}, q}(W) \to \dot{B}_p^{\alpha q}(V)$. If both inequalities hold, $Tr \circ Ext$ is the identity on $\dot{B}_p^{\alpha - \frac{1}{p}, q}(W)$.

1. Introduction

For a point $x \in \mathbb{R}^{n+1}$, we write $x = (x', x_{n+1})$, where $x' = (x_1, x_2, ..., x_n)$. For a continuous function $f : \mathbb{R}^{n+1} \to \mathbb{C}$, we define the trace, or restriction of f, to the hyperplane $\{(x', 0) : x' \in \mathbb{R}^n\}$, which we identify with $\mathbb{R}^n$, by $Tr(f)(x') = f(x', 0)$. It is a classical fact that one can extend the trace operator to be a bounded operator between various classes of functions or distributions on $\mathbb{R}^{n+1}$ whose elements are not necessarily continuous. Such trace theorems are of obvious importance in boundary value problems in partial differential equations. The prototype of a trace theorem is the result of Besov [B] that the restrictions of certain Sobolev spaces are Besov spaces. Denoting the homogeneous Besov spaces on $\mathbb{R}^n$ by $\dot{B}_p^{\alpha q}(\mathbb{R}^n)$, another classical result is that Tr is a bounded operator from $\dot{B}_p^{\alpha q}(\mathbb{R}^{n+1})$ onto $\dot{B}_p^{\alpha - 1/p, q}(\mathbb{R}^n)$, provided $\alpha > \frac{1}{p} + n\left(\frac{1}{p} - 1\right)_+$, for $0 < p, q \le \infty$ (see e.g., [Ta], [T]), where x_+ denotes the maximum of x and 0. We consider the analogous question for matrix-weighted Besov spaces. We find necessary and sufficient conditions on a pair of matrix weights V (on $\mathbb{R}^{n+1}$) and W (on $\mathbb{R}^n$) for $Tr : \dot{B}_p^{\alpha q}(V) \to \dot{B}_p^{\alpha - 1/p, q}(W)$ to be bounded. We also consider a partial inverse operator Ext and find sufficient conditions on V and W for the boundedness of $Ext : \dot{B}_p^{\alpha - 1/p, q}(W) \to \dot{B}_p^{\alpha q}(V)$.

We recall the following definitions from [R1] and [FR]. Let $\mathcal{M}$ be the cone of non-negative-definite $m \times m$ complex-valued matrices. A *matrix weight* W is a locally integrable map $W : \mathbb{R}^n \to \mathcal{M}$. In the scalar case (i.e., $m = 1$), the weight will be

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denoted w. For a measurable vector-valued function $\vec{f}=(f_1,...,f_m)^T:\mathbb{R}^n\to\mathbb{C}^m$, let $\|\vec{f}\|_{L^p(W)}=\left(\int_{\mathbb{R}^n}\|W^{1/p}(t)\vec{f}(t)\|^p\,dt\right)^{1/p}$ for $0<p<\infty$, where $\|\cdot\|$ denotes the usual norm on $\mathbb{C}^m$. We say $W\in A_p$ if $W:\mathbb{R}^n\to\mathcal{M}$ is a.e. invertible, W and W^{-1} are locally integrable, and $\|W\|_{A_p}<\infty$, where, for $1<p<\infty$,

$$\|W\|_{A_p}=\sup_Q\left(\frac{1}{|Q|}\int_Q\left(\frac{1}{|Q|}\int_Q\|W^{1/p}(t)W^{-1/p}(y)\|^p\,dt\right)^{p'/p}dy\right)^{p/p'} \tag{1.1}$$

and, for $0<p\le 1$,

$$\|W\|_{A_p}=\sup_Q\ \operatorname*{ess\,sup}_{y\in Q}\frac{1}{|Q|}\int_Q\left\|W^{1/p}(t)W^{-1/p}(y)\right\|^p dt, \tag{1.2}$$

where the supremum is over all cubes $Q\subseteq\mathbb{R}^n$, $|Q|$ is the Lebesgue measure of Q, $p'=p/(p-1)$ is the conjugate index, and the norm inside the integral is the matrix operator norm. In the scalar weighted case (1.1) reduces to the well-known Muckenhoupt A_p characteristic, see [HMW].

For a cube Q, $\ell(Q)$ denotes the side length of Q. For $c>0$, we let cQ denote the cube with the same center and orientation as Q but whose side length is $c\,\ell(Q)$. A (non-zero) matrix weight W is called a *doubling matrix weight of order* p, $0<p<\infty$, if there exists a constant c such that for all cubes $Q\subseteq\mathbb{R}^n$,

$$\int_{2Q}\|W^{1/p}(t)\,\vec{y}\|^p\,dt\le c\int_Q\|W^{1/p}(t)\,\vec{y}\|^p\,dt, \tag{1.3}$$

i.e., the scalar measures $w_{\vec{y}}(t)=\|W^{1/p}(t)\,\vec{y}\|^p$, for $\vec{y}\in\mathbb{C}^m$, are uniformly doubling. If $c=2^\beta$ is the smallest constant for which (1.3) holds, then β is called the *doubling exponent* of W. An A_p weight is always a doubling weight of order p (see [V] for $p>1$ and [FR, Lemma 2.2] for $p\le 1$).

We say that a matrix weight W is p-admissible if any of the following holds:

(i) $W\in A_p$;

(ii) W is a doubling matrix weight of order p with doubling exponent β, and $p>\beta$;

(iii) $W(t)$ is diagonal for every $t\in\mathbb{R}^n$ and W is a doubling matrix weight of order p.

We say a function $\varphi\in\mathcal{S}(\mathbb{R}^n)$ belongs to the class $\mathcal{A}$ if supp $\hat{\varphi}\subseteq\{\xi\in\mathbb{R}^n:\frac{1}{2}\le|\xi|\le 2\}$ and $|\hat{\varphi}(\xi)|\ge c>0$ if $\frac{3}{5}\le|\xi|\le\frac{5}{3}$. Set $\varphi_\nu(x)=2^{\nu n}\varphi(2^\nu x)$ for $\nu\in\mathbb{Z}$.

Definition 1.1. Suppose $\alpha\in\mathbb{R}$, $0<p<\infty$, $0<q\le\infty$, $\varphi\in\mathcal{A}$ and W is a p-admissible matrix weight on $\mathbb{R}^n$. The Besov space $\dot{B}^{\alpha q}_p(W)$ is the collection of all vector-valued distributions $\vec{f}=(f_1,...,f_m)^T$ with $f_i\in\mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$ (the space of tempered distributions modulo polynomials), $1\le i\le m$, such that

$$\|\vec{f}\|_{\dot{B}^{\alpha q}_p(W)}=\left\|\left\{2^{\nu\alpha}\|\varphi_\nu*\vec{f}\|_{L^p(W)}\right\}_\nu\right\|_{l^q}<\infty,$$

where $\varphi_\nu*\vec{f}=(\varphi_\nu*f_1,...,\varphi_\nu*f_m)^T$.

In [R1] and [FR], it was shown that if W is p-admissible, then $\dot{B}^{\alpha q}_p(W)$ is well-defined, in the sense that two choices of $\varphi\in\mathcal{A}$ yield the same space, with equivalent quasi-norms.

Let $\mathcal{D}_n$ be the collection of dyadic cubes in $\mathbb{R}^n$. Let $\mathcal{D}_{n,\nu} = \{Q \in \mathcal{D}_n : \ell(Q) = 2^{-\nu}\}$. For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $Q_{\nu k}$ denote the dyadic cube $\{(x_1, ..., x_n) \in \mathbb{R}^n : \ k_i \le 2^\nu x_i < k_i + 1, \ i = 1, ..., n\}$ and let $x_Q = 2^{-\nu}k$ be its "lower left corner". For $I \in \mathcal{D}_n$, define $Q(I) \in \mathcal{D}_{n+1}$ by $Q(I) = I \times [0, \ell(I)]$.

Our first result is the following.

Theorem 1.2. *Suppose* $0 < p < \infty$ *and* $0 < q \le \infty$. *Suppose* V *is a* p*-admissible matrix weight on* $\mathbb{R}^{n+1}$ *and* W *is a* p*-admissible matrix weight on* $\mathbb{R}^n$ *with doubling exponent* β. *Assume* $\alpha > \frac{1}{p} + n\left(\frac{1}{p} - 1\right)_+ + \frac{\beta - n}{p}$. *Then the trace operator* Tr *extends to be a continuous map*

$$Tr : \dot{B}_p^{\alpha q}(V) \to \dot{B}_p^{\alpha - \frac{1}{p}, q}(W) \tag{1.4}$$

if and only if there is a constant $C > 0$ *such that*

$$\frac{1}{|I|}\int_I \|W^{1/p}(t)\vec{y}\|^p \, dt \le C \frac{1}{|Q(I)|}\int_{Q(I)} \|V^{1/p}(t)\vec{y}\|^p \, dt \tag{1.5}$$

for all $\vec{y} \in \mathbb{C}^m$ *and all* $I \in \mathcal{D}_n$.

Observe that in the scalar case, (1.5) reduces to

$$\langle w \rangle_I \le c \langle v \rangle_{Q(I)}, \tag{1.6}$$

for all $I \in \mathcal{D}_n$, where $\langle w \rangle_I = \frac{1}{|I|}\int_I w(t)\, dt$ is the average of w over I, and similarly for $\langle v \rangle_{Q(I)}$. In particular, note that (1.6) is independent of p.

In Remark 2.3, we show that we cannot drop the term $(\beta - n)/p$ from the conditions on the indices in Theorem 1.2, even in the scalar case.

We will use Daubechies' wavelets to define an extension operator corresponding to the trace operator. Let K be a positive integer. Let $\{\psi^{(i)}\}_{i=1}^{2^n - 1}$ be the generators of Daubechies' D_L wavelets on $\mathbb{R}^n$ with compact support (see [D]), where L is taken large enough that $\psi^{(i)} \in C^K$. Then $\operatorname{supp} \psi^{(i)} \subseteq r[0,1]^n$, for all i, for some $r > 1$ depending on L. For $I \in \mathcal{D}_n$ and $i = 1, 2, \ldots, 2^n - 1$, define

$$\psi_I^{(i)}(x') = |I|^{-1/2}\psi^{(i)}\left(\frac{x' - x_I}{\ell(I)}\right).$$

Then $\{\psi_I^{(i)}\}_{I \in \mathcal{D}_n, 1 \le i \le 2^n - 1}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. Suppose an integer $\tilde{N}$ to be given. For $t \in \mathbb{R}$, let $h(t)$ be a C^∞ function satisfying $\operatorname{supp} h(t) \subseteq [-1, 1]$, $h(0) = 1$, and $\int_{-1}^1 t^j h(t)\, dt = 0$ for $j = 0, 1, 2, \ldots, \tilde{N}$. The vanishing moment condition on h is regarded as void if $\tilde{N} < 0$. For $I \in \mathcal{D}_n$, $i \in \{1, 2, \ldots, 2^n - 1\}$, and $x = (x', x_{n+1}) \in \mathbb{R}^{n+1}$, let

$$h_{Q(I)}^{(i)}(x) = \psi_I^{(i)}(x') h\left(\frac{x_{n+1}}{\ell(I)}\right).$$

We define

$$Ext(\psi_I^{(i)}) = h_{Q(I)}^{(i)}$$

and extend Ext linearly. We define $Ext(\vec{f}) = (Ext(f_1), \, Ext(f_2), \, \ldots, Ext(f_m))$. Notice that Ext depends on the choice of K and $\tilde{N}$, although we suppress this dependence in the notation. Our main result about the extension operator is the following. Throughout the paper, $[x]$ stands for the greatest integer $\le x$.

Theorem 1.3. *Suppose $0 < p < \infty$, $0 < q \le \infty$, and $\alpha \in \mathbb{R}$. Suppose V is a p-admissible matrix weight on $\mathbb{R}^{n+1}$ with doubling exponent δ, and W is a p-admissible matrix weight on $\mathbb{R}^n$. Define Ext with respect to the integers K and $\tilde{N} = N_v$, where we choose $K > [\alpha]_+ + 1$ and $N_v > \frac{\delta}{p} + (n+1)\left(1 - \frac{1}{p}\right)_+ - n - 1 - \alpha$. Suppose there exists a constant $C > 0$ such that*

$$\frac{1}{|Q(I)|}\int_{Q(I)} \|V^{1/p}(t)\vec{y}\|^p\,dt \le C\frac{1}{|I|}\int_I \|W^{1/p}(t)\vec{y}\|^p\,dt \tag{1.7}$$

for all $I \in \mathcal{D}_n$ and all $\vec{y} \in \mathbb{C}^m$. Then Ext defines a continuous linear map

$$Ext : \dot{B}_p^{\alpha - \frac{1}{p}, q}(W) \to \dot{B}_p^{\alpha q}(V). \tag{1.8}$$

If we also assume (1.5), then $Tr \circ Ext$ is the identity map on $\dot{B}_p^{\alpha - \frac{1}{p}, q}(W)$.

In the scalar case, (1.7) reduces to $\langle v\rangle_{Q(I)} \le c\langle w\rangle_I$ for all $I \in \mathcal{D}_n$. As far as we know, Theorems 1.2 and 1.3 are new even in the scalar case.

The paper is organized as follows: in §2, we prove Theorems 1.2 and 1.3. In §3, we discuss the restriction to planes of codimension higher than 1, which is a routine generalization, and we present a less standard variation in which (1.5) and (1.7) are replaced with versions involving a factor of $\ell(I)^{\kappa p}$, which results in a change in the smoothness index of the Besov space of the restriction. We also remark that our results have obvious analogues for the inhomogeneous Besov spaces $B_p^{\alpha q}(W)$.

Notation: For γ a multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$ with $\gamma_i \in \mathbb{N} \cup \{0\}$, $1 \le i \le n$, the standard multi-index notation is used, such as $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$, and $D^\gamma = \partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\cdots\partial_{x_n}^{\gamma_n}$. In general $A \approx B$ means that there exist positive constants c_1, c_2 such that $c_1 B \le A \le c_2 B$.

2. Proofs of the Main Results

We recall the definition of reducing operators associated to matrix weight (see [V], and, for the case $p < 1$, [FR]). For a matrix weight W on $\mathbb{R}^n$ and $0 < p < \infty$, there exists a sequence of positive definite matrices $\{A_{Q,W,p}\}_{Q \in \mathcal{D}_n}$, called reducing operators for W, such that for all $\vec{y} \in \mathbb{C}^m$,

$$\|A_{Q,W,p}\,\vec{y}\,\| \approx \left(\frac{1}{|Q|}\int_Q \|W^{1/p}(t)\vec{y}\,\|^p\,dt\right)^{1/p},$$

with equivalence constants independent of Q and $\vec{y}$. The sequence of reducing operators is not unique, but we suppose one to be chosen.

We note that condition (1.5) can be written in terms of reducing operators for V and W as

$$\|A_{I,W,p}\,\vec{y}\,\| \le C\|A_{Q(I),V,p}\,\vec{y}\,\| \tag{2.1}$$

for all $\vec{y} \in \mathbb{C}^m$ and all $I \in \mathcal{D}_n$.

Our analysis depends crucially on the relation between the space $\dot{B}_p^{\alpha q}(W)$ and its discrete analogue $\dot{b}_p^{\alpha q}(W)$, defined as follows.

Definition 2.1. Suppose $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$ and W is a p-admissible matrix weight on $\mathbb{R}^n$. The space $\dot{b}_p^{\alpha q}(W)$ consists of all vector-valued sequences

$\vec{s} = \{\vec{s}_Q\}_{Q\in\mathcal{D}_n}$ such that

$$\|\vec{s}\|_{\dot{b}_p^{\alpha q}(W)} = \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{Q\in\mathcal{D}_{n,\nu}} |Q|^{-\frac{1}{2}} \vec{s}_Q \chi_Q \right\|_{L^p(W)} \right\}_\nu \right\|_{l^q} < \infty.$$

By carrying out the integration in Definition 2.1, we see that

(2.2)
$$\|\vec{s}\|_{\dot{b}_p^{\alpha q}(W)} = \left(\sum_{\nu\in\mathbb{Z}} \left(\sum_{Q\in\mathcal{D}_{n,\nu}} \ell(Q)^{-\alpha p+n-np/2} \frac{1}{|Q|} \int_Q \|W^{1/p}(t)\vec{s}_Q\|^p \, dt \right)^{q/p} \right)^{1/q}$$

$$\approx \left(\sum_{\nu\in\mathbb{Z}} \left(\sum_{Q\in\mathcal{D}_{n,\nu}} \ell(Q)^{-\alpha p+n-np/2} \|A_{Q,W,p}\vec{s}_Q\|^p \right)^{q/p} \right)^{1/q}. \tag{2.3}$$

The sequence space $\dot{b}_p^{\alpha q}(W)$ corresponds to $\dot{B}_p^{\alpha q}(W)$ under the wavelet representation, in the following sense. Let $\{\psi^{(i)}\}_{i=1}^{2^n-1}$ be Daubechies' D_L wavelets on $\mathbb{R}^n$ with compact support (see [D]), chosen with L sufficiently large so that each $\psi^{(i)} \in C^K$ with $K \geq [\alpha]_+ + 1$ and each $\psi^{(i)}$ has sufficiently many vanishing moments (the number depending on α, p, and q). Note that each $\psi_I^{(i)}$ satisfies $\operatorname{supp} \psi_I^{(i)} \subseteq rI$ and $\psi_I^{(i)} \in C^K$ with

$$|D^\gamma \psi_I^{(i)}(x')| \leq C|I|^{-1/2-|\gamma|/n}, \tag{2.4}$$

for all $|\gamma| \leq K$ and all $x' \in \mathbb{R}^n$. Then by [R1, Corollary 10.3] and [FR, Theorem 4.4], every $\vec{f} \in \dot{B}_p^{\alpha q}(W)$ can be written as $\vec{f} = \sum_{i=1}^{2^n-1} \sum_{I\in\mathcal{D}^n} \langle \vec{f}, \psi_I^{(i)} \rangle \psi_I^{(i)}$ with convergence of the sum in $\dot{B}_p^{\alpha q}(W)$-quasinorm if $q < \infty$, and in $\mathcal{S}'/\mathcal{P}$ if $q = \infty$, and

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} \approx \sum_{i=1}^{2^n-1} \|\{\langle \vec{f}, \psi_I^{(i)} \rangle\}_{I\in\mathcal{D}_n}\|_{\dot{b}_p^{\alpha q}(W)}. \tag{2.5}$$

One direction of the equivalence (2.5) holds under the much more general conditions that the terms in the sum are "smooth molecules" for $\dot{B}_p^{\alpha q}(W)$. We state a simplified version, which will be sufficient for our purposes, of the molecular decomposition theorem for Besov spaces. In a more general form, this result is proved for $p > 1$ in [R1, Theorem 5.2], and for $0 < p \leq 1$ in [FR, Theorem 3.1].

Theorem 2.2. *Suppose $\alpha \in \mathbb{R}, 0 < p < \infty, 0 < q \leq \infty$ and let W be a p-admissible matrix weight on $\mathbb{R}^n$ with doubling exponent β. Let $N = \max([\frac{\beta}{p} + n(1-\frac{1}{p})_+ - n - \alpha], -1)$ and suppose $M > \frac{\beta}{p} + n(1-\frac{1}{p})_+$. Suppose $\{m_Q\}_{Q\in\mathcal{D}_n}$ is a family of functions satisfying*

(M1) $\int x^\gamma m_Q(x)\, dx = 0,$ *for* $|\gamma| \leq N,$

(M2) $|m_Q(x)| \leq |Q|^{-1/2} \left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-\max(M, M-\alpha)},$

and

(M3) $|D^\gamma m_Q(x)| \le |Q|^{-1/2-|\gamma|/n}\left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-M}$ *if* $|\gamma| \le [\alpha]+1$,

for each $Q\in\mathcal{D}_n$. *Suppose* $\vec{s}=\{\vec{s}_Q\}_{Q\in\mathcal{D}_n}\in \dot{b}_p^{\alpha q}(W)$. *Then* $\sum_{Q\in\mathcal{D}_n}\vec{s}_Q\, m_Q\in \dot{B}_p^{\alpha q}(W)$, *and there exists* c *independent of* $\vec{s}$ *and* $\{m_Q\}_Q$ *such that*

$$\left\|\sum_{Q\in\mathcal{D}_n}\vec{s}_Q\, m_Q\right\|_{\dot{B}_p^{\alpha q}(W)} \le c\,\|\vec{s}\,\|_{\dot{b}_p^{\alpha q}(W)}. \tag{2.6}$$

It is understood that (M1) is void if $N=-1$. We now prove Theorem 1.2.

Proof (of Theorem 1.2). We first show the necessity of (1.5). Suppose (1.4). Fix $I\in\mathcal{D}_n$ and $\vec{y}\in\mathbb{C}^m$. Define the sequence $\vec{t}=\{\vec{t}_J\}_{J\in\mathcal{D}_n}$ by letting $\vec{t}_J=0$ if $J\neq I$, and $\vec{t}_I=\ell(I)^{\alpha-\frac{n+1}{p}+\frac{n}{2}}\vec{y}$. Let

$$\vec{g}(x')=\vec{t}_I\psi_I^{(1)}(x').$$

Then by (2.5),

$$\|\vec{g}\,\|^p_{\dot{B}_p^{\alpha-\frac{1}{p},q}(W)} \approx \|\vec{t}\,\|^p_{\dot{b}_p^{\alpha-\frac{1}{p},q}(W)} = \ell(I)^{-(\alpha-\frac{1}{p})p+n-np/2}\|A_{I,W,p}\,\vec{t}_I\|^p=\|A_{I,W,p}\vec{y}\,\|^p. \tag{2.7}$$

Define Ext with respect to integers $K\ge[\alpha]_+ +1$, and $\tilde{N}=N_v=\frac{\beta}{p}+(n+1)\left(1-\frac{1}{p}\right)_+ -n-1-\alpha$. Let

$$\vec{f}(x)=Ext(\vec{g}\,)(x)=\vec{t}_I\psi_I^{(1)}(x')h\left(\frac{x_{n+1}}{\ell(I)}\right).$$

Then $\vec{f}\in C^K$ and

$$Tr(\vec{f}\,)(x')=\vec{f}(x',0)=\vec{g}(x')$$

by the assumption $h(0)=1$. Define $\vec{s}=\{\vec{s}_Q\}_{Q\in\mathcal{D}_{n+1}}$ by letting $\vec{s}_Q=0$ for $Q\neq Q(I)$, and $\vec{s}_{Q(I)}=\ell(I)^{1/2}\vec{t}_I$. Let

$$m_{Q(I)}(x)=\ell(I)^{-(n+1)/2}\psi^{(1)}\left(\frac{x'-x_I}{\ell(I)}\right)h\left(\frac{x_{n+1}}{\ell(I)}\right)=\ell(I)^{-1/2}\psi_I(x')h\left(\frac{x_{n+1}}{\ell(I)}\right).$$

Then $\vec{f}=\vec{s}_{Q(I)}m_{Q(I)}$. Note that $\operatorname{supp} m_{Q(I)}\subseteq rQ(I)$,

$$|D^\gamma m_{Q(I)}(x)|\le C\ell(I)^{-(n+1)/2-|\gamma|}=C|Q(I)|^{-\frac{1}{2}-\frac{|\gamma|}{n+1}}$$

for $|\gamma|\le K$, and

$$\int x^\gamma m_{Q(I)}(x)\,dx=0$$

for $0\le|\gamma|\le N_v$, by the vanishing moment condition for h. Hence, up to a multiplicative constant, $m_{Q(I)}$ is a smooth molecule for $Q(I)$ for $\dot{B}_p^{\alpha q}(V)$. Therefore, by Theorem 2.2,

$$\begin{aligned}\|\vec{f}\,\|^p_{\dot{B}_p^{\alpha q}(V)} \le c\|\vec{s}\,\|^p_{\dot{b}_p^{\alpha q}(V)} &= c\,\ell(Q(I))^{-\alpha p+n+1-(n+1)p/2}\|A_{Q(I),V,p}\,\vec{s}_{Q(I)}\,\|^p \\ &= c\,\|A_{Q(I),V,p}\,\vec{y}\,\|^p.\end{aligned} \tag{2.8}$$

Using (2.7), our assumption (1.4), and then (2.8), we obtain

$$\|A_{I,W,p}\vec{y}\,\|\le c\,\|\vec{g}\,\|_{\dot{B}_p^{\alpha-\frac{1}{p},q}(W)}\le c\,\|\vec{f}\,\|_{\dot{B}_p^{\alpha q}(V)}\le c\,\|A_{Q(I),V,p}\,\vec{y}\,\|.$$

Now suppose (1.5) holds for all $I \in \mathcal{D}_n$, and let $\vec{f} \in \dot{B}_p^{\alpha q}(V)$. Define coefficient sequences $\vec{s}^{(i)} = \{\vec{s}_Q^{(i)}\}_{Q\in\mathcal{D}_{n+1}}$ by $\vec{s}_Q^{(i)} = \langle f, \psi_Q^{(i)}\rangle$, where $\{\psi_Q^{(i)}\}_{Q\in\mathcal{D}_{n+1}, 1\le i\le 2^{n+1}-1}$ are Daubechies' wavelets on $\mathbb{R}^{n+1}$ as above for $K > [\alpha]_+ + 1$. Then

$$\vec{f} = \sum_{i=1}^{2^{n+1}-1} \sum_{Q\in\mathcal{D}_{n+1}} \vec{s}_Q^{\,(i)} \psi_Q^{(i)}, \tag{2.9}$$

and $\|\vec{s}\|_{\dot{b}_p^{\alpha q}(V)} \le C\, \|\vec{f}\,\|_{\dot{B}_p^{\alpha q}(V)}$, by (2.5).

If the sum in (2.9) is finite, then $Tr(\vec{f}\,)$ is defined, since each $\psi_Q^{(i)} \in C^1$. For $q < \infty$, if we establish the estimate $\|Tr(\vec{f}\,)\|_{\dot{B}_p^{\alpha-\frac{1}{p},q}(W)} \le c\|\vec{f}\,\|_{\dot{B}_p^{\alpha q}(V)}$ for such $\vec{f}$, then we can extend Tr to be a bounded, continuous map from all of $\dot{B}_p^{\alpha q}(V)$ into $\dot{B}_p^{\alpha-\frac{1}{p},q}(W)$, since the quasi-norm convergence of the wavelet expansion shows that the finite sums of the form in (2.9) are dense in $\dot{B}_p^{\alpha q}(V)$. Once this is done, for $\vec{f} \in \dot{B}_p^{\alpha\,\infty}(V)$, let $\vec{g} = \sum_{i=1}^{2^{n+1}-1} \sum_{\substack{Q\,\in\,\mathcal{D}_{n+1},\\ \ell(Q)\,\le\,1}} \vec{s}_Q^{\,(i)}\psi_Q^{(i)}$ and $\vec{h} = \sum_{i=1}^{2^{n+1}-1} \sum_{\substack{Q\,\in\,\mathcal{D}_{n+1},\\ \ell(Q)\,>\,1}} \vec{s}_Q^{\,(i)}\psi_Q^{(i)}$. Pick $\alpha_0 < \alpha < \alpha_1$, where $\alpha_0 > \frac{1}{p} + n\left(\frac{1}{p} - 1\right)_+ + \frac{\beta-n}{p}$. Note that $\vec{g} \in \dot{B}_p^{\alpha_0,1}(V)$ and $\vec{h} \in \dot{B}_p^{\alpha_1,1}(V)$. Hence, by the results for $q < \infty$, $Tr(\vec{g})(x') = \sum_{i=1}^{2^{n+1}-1} \sum_{\substack{Q\,\in\,\mathcal{D}_{n+1},\\ \ell(Q)\,\le\,1}} \vec{s}_Q^{\,(i)}\psi_Q^{(i)}(x',0)$ and $Tr(\vec{h})(x') = \sum_{i=1}^{2^{n+1}-1} \sum_{\substack{Q\,\in\,\mathcal{D}_{n+1},\\ \ell(Q)\,>\,1}} \vec{s}_Q^{\,(i)}\psi_Q^{(i)}(x',0)$ exist in $\dot{B}_p^{\alpha_0-1/p,1}(W)$ and $\dot{B}_p^{\alpha_1-1/p,1}(W)$, respectively, and hence, in $\mathcal{S}'/\mathcal{P}$. Therefore, $Tr(\vec{f}\,)(x') = Tr(\vec{g})(x') + Tr(\vec{h})(x') = \sum_{i=1}^{2^{n+1}-1} \sum_{Q\in\mathcal{D}_{n+1}} \vec{s}_Q^{\,(i)}\psi_Q^{(i)}(x',0)$ exists in $\mathcal{S}'/\mathcal{P}$.

Recall that $\operatorname{supp} \psi_Q^{(i)} \subseteq rQ$. Let J be the smallest integer such that $2J+1 \ge r$. If $Q = Q_{\nu,k}$, where $k = (k', j)$ (here $k \in \mathbb{Z}^{n+1}$ and $k' = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$), and $j \notin \{-J, -J+1, \ldots, J-1\}$, then $Tr(\psi_Q^{(i)})(x') = \psi_Q^{(i)}(x',0) = 0$. Therefore,

$$Tr\vec{f}\,(x') = \sum_{i=1}^{2^{n+1}-1} \sum_{\nu\in\mathbb{Z}} \sum_{k'\in\mathbb{Z}^n} \sum_{j=-J}^{J-1} \vec{s}^{\,(i)}_{Q_{\nu,(k',j)}} \psi^{(i)}_{Q_{\nu,(k',j)}}(x',0).$$

For $-J \le j \le J-1$ and $I \in \mathcal{D}_n$, let $Q^{(j)}(I) = I \times [j\ell(Q), (j+1)\ell(Q)]$. Then

$$Tr(\vec{f}\,) = \sum_{i=1}^{2^{n+1}-1} \sum_{j=-J}^{J-1} Tr^{(i,j)}(\vec{f}\,),$$

where

$$Tr^{(i,j)}(\vec{f}\,)(x') = \sum_{\nu\in\mathbb{Z}} \sum_{k'\in\mathbb{Z}^n} \vec{s}^{\,(i)}_{Q_{\nu,(k',j)}} \psi^{(i)}_{Q_{\nu,(k',j)}}(x',0) = \sum_{I\in\mathcal{D}_n} \vec{s}^{\,(i)}_{Q^{(j)}(I)} \psi^{(i)}_{Q^{(j)}(I)}.$$

Notice that

$$\|A_{I,W,p}\,\vec{y}\,\| \le C\, \|A_{Q^{(j)}(I),V,p}\,\vec{y}\,\|,$$

for $-J \le j \le J-1$, with constants depending only on J, by (1.5) and the doubling property for V, since

$$\|A_{Q(I),V,p}\vec{y}\,\| \le C\|A_{(2J+1)Q(I),V,p}\vec{y}\,\| \le C\|A_{Q^{(j)}(I),V,p}\vec{y}\,\|.$$

Using this, all of the terms $Tr^{(i,j)}(\vec{f})$ can be estimated in the same way, so for the remainder of the proof we drop the indices i and j. Hence, we consider $Tr(\vec{f}) = \sum_{I\in\mathcal{D}_n} \vec{s}_{Q(I)}\psi_{Q(I)}$.

For $I \in \mathcal{D}_n$, let $\vec{t}_I = \ell(I)^{-1/2}\vec{s}_{Q(I)}$ and $b_I(x') = \ell(I)^{1/2}\psi_{Q(I)}(x', 0)$. We obtain

$$Tr(\vec{f}) = \sum_{I\in\mathcal{D}_n} \vec{t}_I b_I,$$

where each b_I satisfies $\operatorname{supp} b_I \subseteq rI$ and

$$|D^\gamma b_I(x')| \le c_\gamma\, \ell(I)^{1/2}\ell(Q(I))^{-\frac{n+1}{2}-|\gamma|} = c_\gamma\, \ell(I)^{-\frac{n}{2}-|\gamma|} = c_\gamma\, |I|^{-\frac{1}{2}-\frac{|\gamma|}{n}},$$

for every multi-index γ, by (2.4), since $\ell(Q(I)) = \ell(I)$. Since $\alpha > \frac{1}{p} + n\left(\frac{1}{p}-1\right)_+ + \frac{\beta-n}{p} = \frac{1}{p} + \frac{\beta}{p} + n\left(1-\frac{1}{p}\right)_+ - n$, the vanishing moment condition in Theorem 2.2 is void for $\dot{B}_p^{\alpha-\frac{1}{p},q}(W)$. Hence, Theorem 2.2 implies

$$\|Tr(\vec{f})\|_{\dot{B}_p^{\alpha-\frac{1}{p},q}(W)} \le c\,\|\vec{t}\|_{\dot{b}_p^{\alpha-\frac{1}{p},q}(W)}$$

$$= \left(\sum_{\nu\in\mathbb{Z}}\left(\sum_{I\in\mathcal{D}_{n,\nu}} \ell(I)^{-(\alpha-1/p)p+n-np/2}\|A_{I,W,p}\vec{t}_I\|^p\right)^{q/p}\right)^{1/q},$$

by (2.3), where the sum on ν is replaced by the supremum over ν if $q = \infty$. Using (1.5) (equivalently, (2.1)), recalling that $\vec{t}_I = \ell(I)^{-1/2}\vec{s}_{Q(I)}$, and regarding the sum over $I \in \mathcal{D}_{n,\nu}$ as a sum over $\{Q(I), I \in \mathcal{D}_{n,\nu}\} \subseteq D_{n+1,\nu}$, we obtain

$$\|Tr(\vec{f})\|_{\dot{B}_p^{\alpha-\frac{1}{p},q}(W)} \le c\left(\sum_{\nu\in\mathbb{Z}}\left(\sum_{Q\in\mathcal{D}_{n+1,\nu}} \ell(Q)^{-\alpha p+n+1-(n+1)p/2}\|A_{Q(I),V,p}\,\vec{s}_Q\,\|^p\right)^{q/p}\right)^{1/q}$$

$$= c\,\|\vec{s}\|_{\dot{b}_p^{\alpha q}(V)} \le c\|\vec{f}\|_{\dot{B}_p^{\alpha q}(V)}.$$

□

Remark 2.3. In the classical scalar, unweighted case, the trace operator $Tr : \dot{B}_p^{\alpha q}(\mathbb{R}^{n+1}) \to \dot{B}_p^{\alpha-\frac{1}{p},q}(\mathbb{R}^n)$ is bounded for $\alpha > 1/p + n(1/p-1)_+$, whereas in Theorem 1.2 the range of indices is $\alpha > 1/p + n(1/p-1)_+ + (\beta-n)/p$, where β is the doubling exponent of w. If v and w are Lebesgue measure, then $\beta = n$ and we recover the classical result. One may think that the term $(\beta-n)/p$ may be dropped in Theorem 1.2, but this is not the case. Even in the scalar case, the need for some dependence on β can be easily seen. For $r \ge 0$, let $w(x') = |x'|^r$ be a scalar weight on $\mathbb{R}^n$ and let $v(x) = v(x', x_{n+1}) = w(x')$; that is, v is just w multiplied by Lebesgue measure in the x_{n+1} direction. Then $|I|^{-1}\int_I w = |Q(I)|^{-1}\int_{Q(I)} v$, which gives (1.5) in the scalar case. Also w is doubling with doubling exponent $\beta = r+n$, and v is doubling, so both v and w are p-admissible for the scalar case. However, we claim that $Tr : \dot{B}_p^{\alpha q}(v) \to \dot{B}_p^{\alpha-\frac{1}{p},q}(w)$ fails for $\alpha \le 1/p + n(1/p-1) + (\beta-n)/p$, for $0 < p < \infty$. To see this, let $\gamma \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\hat{\gamma}(\xi) = 0$ for $|\xi| > 4$ and $\hat{\gamma} = 1$ for $|\xi| \le 2$. Let $h : \mathbb{R} \to \mathbb{R}$ be C^∞ and satisfy (as in the definition of Ext) $\operatorname{supp} h \subseteq [-1, 1]$, $h(0) = 1$, and $\int t^j h(t)\,dt = 0$ for $0 \le j \le M$, where M is taken large enough that the conditions of Theorem 2.2 apply to v. Let

$f(x) = \gamma(x')h(x_{n+1})$. Then, up to a constant factor, f satisfies the molecular conditions of (M1), (M2), and (M3) of Theorem 2.2 for $Q = Q_{00}$. Thus, $f \in \dot{B}_p^{\alpha q}(v)$. Moreover, $Tr(f) = \gamma$, since $h(0) = 1$. But for $\nu \le 0$, $\gamma * \varphi_\nu = \varphi_\nu$, because $\hat{\gamma} = 1$ on supp $\hat{\varphi}_\nu$. Hence, for $\nu \le 0$,

$$\|\gamma * \varphi_\nu\|_{L^p(w)}^p = \int_{\mathbb{R}^n} |2^{\nu n}\varphi(2^\nu x)|^p |x|^r\, dx = c_{r,p} 2^{\nu(np-n-r)},$$

where $c_{r,p} = \int |\varphi(x)|^p |x|^r\, dx$. Recalling that $r = \beta - n$, we have

$$\|Tr(f)\|_{\dot{B}_p^{\alpha-1/p,q}(w)}^q \ge \sum_{\nu=-\infty}^{0} \left(2^{\nu(\alpha-1/p)} \|\gamma * \varphi_\nu\|_{L^p(w)}\right)^q$$

$$= c_{r,p}^{q/p} \sum_{\nu=-\infty}^{0} 2^{\nu(\alpha-1/p+n-n/p-r/p)q} = \infty,$$

for $\alpha \le 1/p + n(1/p-1) + r/p$, and therefore, $Tr(f) \notin \dot{B}_p^{\alpha-1/p,q}(w)$, as claimed. For $0 < p \le 1$, this observation shows that the indices in Theorem 1.2 cannot be improved in general. For $p > 1$, the overlap of supports of the molecules is critical in determining the Besov norm (see [FJ1, Theorem 3.1]), so a sharp example based on a single molecule is not possible. However, even for $p > 1$, as $r \to \infty$, the observation above shows that we cannot obtain the trace imbedding for all $\alpha > 1/p$ as in the unweighted scalar case.

Here is the proof of Theorem 1.3.

Proof (of Theorem 1.3). Note that (1.7) is equivalent to

$$\|A_{Q(I),V,p}\,\vec{y}\,\| \le C\|A_{I,W,p}\,\vec{y}\,\|. \tag{2.10}$$

Suppose (1.7) holds and let $\vec{f} \in \dot{B}_p^{\alpha-\frac{1}{p},q}(W)$. Then $\vec{f} = \sum_{i=1}^{2^n-1} \sum_{I\in\mathcal{D}_n} \vec{t}_I^{\,(i)} \psi_I^{(i)}$, where $\vec{t}_I^{\,(i)} = \langle \vec{f}, \psi_I^{(i)} \rangle$. Let $\vec{t}^{\,(i)} = \{\vec{t}_I^{\,(i)}\}_{I\in\mathcal{D}_n}$ for each $i = 1, 2, \ldots, 2^n - 1$. Also, for each i, define a sequence $\vec{s}^{\,(i)} = \{\vec{s}_Q^{\,(i)}\}_{Q\in\mathcal{D}_{n+1}}$ by setting $\vec{s}_{Q(I)}^{\,(i)} = \ell(I)^{1/2}\vec{t}_I^{\,(i)}$ when $Q = Q(I)$ for some $I \in \mathcal{D}_n$ and $\vec{s}_Q^{\,(i)} = 0$ if Q is not of the form $Q(I)$. Furthermore, define $m_Q^{(i)}$ by setting $m_{Q(I)}^{(i)} = \ell(I)^{-1/2} Ext(\psi_I^{(i)})$ for $Q = Q(I), I \in \mathcal{D}$, and $m_Q^{(i)} = 0$ otherwise. Then, by definition,

$$Ext(\vec{f}\,) = \sum_{i=1}^{2^n-1} \sum_{I\in\mathcal{D}_n} \vec{t}_I^{\,(i)} Ext(\psi_I^{(i)}) = \sum_{i=1}^{2^n-1} \sum_{Q\in\mathcal{D}_{n+1}} \vec{s}_Q^{\,(i)} m_Q^{(i)}.$$

It follows that Ext is linear. Also, as in the previous argument, for a small enough constant c, each $c\,m_Q^{(i)}$ is a smooth molecule for $\dot{B}_p^{\alpha q}(V)$. Hence, by Theorem 2.2,

$$\|Ext(\vec{f}\,)\|_{\dot{B}_p^{\alpha q}(V)} \le c \sum_{i=1}^{2^n-1} \|\vec{s}^{\,(i)}\|_{\dot{b}_p^{\alpha q}(V)}$$

$$= \sum_{i=1}^{2^n-1} \left(\sum_{\nu\in\mathbb{Z}} \left(\sum_{Q\in\mathcal{D}_{n+1,\nu}} \ell(Q)^{-\alpha p+n+1-(n+1)p/2} \|A_{Q,V,p}\,\vec{s}_Q^{\,(i)}\,\|^p \right)^{q/p} \right)^{1/p}$$

$$= \sum_{i=1}^{2^n-1} \left(\sum_{\nu\in\mathbb{Z}} \left(\sum_{I\in\mathcal{D}_{n,\nu}} \ell(I)^{-(\alpha-1/p)p+n-np/2} \|A_{Q(I),V,p}\, \vec{t}_I^{\,(i)}\,\|^p \right)^{q/p} \right)^{1/p},$$

using the definition of $\vec{s}_Q^{\,(i)}$. By the assumption (1.7), we obtain

$$\|Ext(\vec{f})\|_{\dot{B}_p^{\alpha q}(V)} \leq c \sum_{i=1}^{2^n-1} \|\vec{t}^{\,(i)}\|_{\dot{b}_p^{\alpha-\frac{1}{p},q}(W)} \leq c\|\vec{f}\|_{\dot{B}_p^{\alpha-\frac{1}{p},q}(W)},$$

by (2.5). This proves the continuity of Ext on $\dot{B}_p^{\alpha-\frac{1}{p},q}(W)$.

Note that $Tr \circ Ext$ is the identity on finite sums of the form $\sum_{i=1}^{2^n-1} \sum_{I\in\mathcal{D}_n} \vec{t}_I^{\,(i)} \psi_I^{(i)}$, by the wavelet identity. Thus, for $q < \infty$, the density of these finite sums in $\dot{B}_p^{\alpha-\frac{1}{p},q}(W)$ and the continuity of Tr and Ext imply that $Tr \circ Ext$ is the identity on $\dot{B}_p^{\alpha-\frac{1}{p},q}(W)$. This result and the imbedding $\dot{B}_p^{\alpha-1/p,\infty}(W) \to \dot{B}_p^{\alpha_0-1/p,1}(W) + \dot{B}_p^{\alpha_1-1/p,1}(W)$, where $\alpha_0 < \alpha < \alpha_1$, guarantee that $Tr \circ Ext$ is the identity on $\dot{B}_p^{\alpha-\frac{1}{p},\infty}(W)$. □

3. Variations and Generalizations

We consider generalizations of Theorems 1.2 and 1.3 in two directions. The first direction is standard, in which we consider the trace when the change in dimensions could be greater than 1. The second is not standard. It involves modifying the estimates (1.5) and (1.7) with factors of $\ell(I)$ to a power, which results in a change of the smoothness index in the Besov space of the restriction. For maximum concision and generality, we combine these two generalizations in the following statements.

In this section, n is a positive integer and j is a positive integer with $j \leq n$. For $x \in \mathbb{R}^{n+1}$, we denote $x = (x', x'')$, where $x' \in \mathbb{R}^j$ and $x'' \in \mathbb{R}^{n+1-j}$. For a continuous function $f : \mathbb{R}^{n+1} \to \mathbb{C}$, we define $Tr(f) : \mathbb{R}^j \to \mathbb{C}$, the trace of f, by $Trf(x') = f(x', 0)$. For $I \in \mathcal{D}_j$, we define $Q(I) = I \times [0, \ell(I)]^{n+1-j} \in \mathcal{D}_{n+1}$.

Theorem 3.1. *Suppose* $0 < p < \infty$ *and* $0 < q \leq \infty$, $\kappa \in \mathbb{R}$. *Suppose* V *is a* p*-admissible matrix weight on* $\mathbb{R}^{n+1}$ *and* W *is a* p*-admissible matrix weight on* $\mathbb{R}^j$ *with doubling exponent* β. *Assume* $\alpha > \frac{n+1-j}{p} + j\left(\frac{1}{p} - 1\right)_+ + \frac{\beta-j}{p}$. *Then the trace operator* Tr *extends to be a continuous map*

$$Tr : \dot{B}_p^{\alpha q}(V) \to \dot{B}_p^{\alpha-\frac{n+1-j}{p}+\kappa,\, q}(W) \tag{3.1}$$

if and only if there is a constant $C > 0$ *such that*

$$\frac{1}{|I|}\int_I \|W^{1/p}(t)\vec{y}\|^p\,dt \leq C\,[l(I)]^{\kappa p}\,\frac{1}{|Q(I)|}\int_{Q(I)} \|V^{1/p}(t)\vec{y}\|^p\,dt \tag{3.2}$$

for all $\vec{y} \in \mathbb{C}^m$ *and all* $I \in \mathcal{D}_j$.

To define the extension operator, let $\{\psi^{(i)}\}_{i=1}^{2^j-1}$ be the generators of Daubechies' D_L wavelets on $\mathbb{R}^j$, for L large enough. Let $\psi_I^{(i)}$ and h be as in the definition of Ext in §1. Define $Ext(\psi_I^{(i)}) : \mathbb{R}^{n+1} \to \mathbb{C}$ by

$$Ext(\psi_I^{(i)})(x) = (\psi_I^{(i)})(x')h\left(\frac{x_{j+1}}{\ell(I)}\right)h\left(\frac{x_{j+2}}{\ell(I)}\right)\cdots h\left(\frac{x_{n+1}}{\ell(I)}\right).$$

We extend Ext by linearity.

Theorem 3.2. *Suppose $0 < p < \infty$, $0 < q \leq \infty$, $\alpha, \kappa \in \mathbb{R}$. Suppose V is a p-admissible matrix weight on $\mathbb{R}^{n+1}$ with doubling exponent δ, and W is a p-admissible matrix weight on $\mathbb{R}^j$. Define Ext with respect to the integers K and $\tilde{N} = N_v$, where we choose $K > [\alpha]_+ + 1$ and $N_v > \frac{\delta}{p} + (n+1)\left(1 - \frac{1}{p}\right)_+ - n - 1 - \alpha$. Suppose there exists a constant $C > 0$ such that*

$$\frac{1}{|Q(I)|} \int_{Q(I)} \|V^{1/p}(t)\vec{y}\|^p \, dt \leq C\, [l(I)]^{-\kappa p} \frac{1}{|I|} \int_I \|W^{1/p}(t)\vec{y}\|^p \, dt \tag{3.3}$$

for all $I \in \mathcal{D}_j$ and all $\vec{y} \in \mathbb{C}^m$. Then Ext defines a continuous linear map

$$Ext : \dot{B}_p^{\alpha - \frac{n+1-j}{p} + \kappa,\, q}(W) \to \dot{B}_p^{\alpha q}(V). \tag{3.4}$$

If (3.2) also holds, then $Tr \circ Ext$ is the identity operator on $\dot{B}_p^{\alpha - \frac{n+1-j}{p} + \kappa,\, q}(W)$.

The case $\kappa = 0$ gives the matrix-weight analogue of the standard restriction results from $\mathbb{R}^{n+1}$ to $\mathbb{R}^j$; if we take $j = n$ we have Theorems 1.2 and 1.3.

To see how the case $\kappa \neq 0$ can arise, suppose $j = n$ and let $r > -1$. Consider the scalar weights v and w, where w is Lebesgue measure on $\mathbb{R}^n$ and v on $\mathbb{R}^{n+1}$ is defined by $v(x) = |x_{n+1}|^r$. Both are doubling weights, and hence are p-admissible (because they are scalar weights). For $I \in \mathcal{D}_j$,

$$\frac{1}{|Q(I)|} \int_{Q(I)} v(t)\, dt = \frac{1}{r+1} \ell(I)^r = \frac{1}{r+1} \ell(I)^r \frac{1}{|I|} \int_I w(t)\, dt.$$

Thus, we have conditions (3.2) and (3.3) with $\kappa = -r/p$. It may seem strange that one obtains the trace imbedding with a different smoothness index. However, the Besov norm measures smoothness in the weighted L^p norm. If the weight blows up or decays to 0 as the hyperplane $\mathbb{R}^j$ is approached, the finiteness of the Besov norm on $\mathbb{R}^{n+1}$ places stronger or weaker conditions on the smoothness of the function on the hyperplane. This is reflected in the behavior of the coefficients in the wavelet expansion corresponding to the small dyadic cubes near the restriction plane.

The proofs of Theorems 3.1 and 3.2 are simple modifications of the proofs of Theorems 1.2 and 1.3. We make only a few comments on the changes needed. Observe that condition (3.2) is equivalent to

$$\|A_{I,W,p}\, \vec{y}\,\| \leq C\, [l(I)]^{\kappa}\, \|A_{Q(I),V,p}\, \vec{y}\,\| \tag{3.5}$$

for all $\vec{y} \in \mathbb{C}^m$ and all $I \in \mathcal{D}_j$. To show the necessity of (3.2) in Theorem 3.1, fix $I \in \mathcal{D}_j$ and $\vec{y} \in \mathbb{C}^m$. Define $\vec{t} = \{\vec{t}_J\}_{J \in \mathcal{D}_j}$ by letting $\vec{t}_J = 0$ if $J \neq I$, and $\vec{t}_I = \ell(I)^{\alpha - \frac{n+1}{p} + \kappa + \frac{j}{2}} \vec{y}$. Let $\vec{g}(x') = \vec{t}_I \psi_I^{(1)}(x')$. Then $\|\vec{g}\,\|_{\dot{B}_p^{\alpha - \frac{n+1-j}{p} + \kappa,\, q}(W)} \approx \|A_{I,W,p}\vec{y}\,\|$. Let $\vec{f} = \vec{t}_I Ext(\psi_I^{(1)}) = \vec{s}_{Q(I)} m_{Q(I)}$, where $\vec{s}_{Q(I)} = \ell(I)^{(n+1-j)/2} \vec{t}_I$ and $m_{Q(I)} = \ell(I)^{-(n+1-j)/2} Ext(\psi_I^{(1)})$. Then $Tr(\vec{f}) = \vec{g}$. Letting $\vec{s}_Q = 0$ for Q not of the form $Q(I)$, we obtain $\|\vec{f}\|_{\dot{B}_p^{\alpha q}(V)} \leq C \ell(I)^{\kappa} \|A_{Q(I),V,p}\vec{y}\,\|$. Hence, the boundedness of Tr implies (3.5), and thus, (3.2).

Now assume (3.2) holds. We can suppose $\vec{f} = \sum_{I \in \mathcal{D}_j} \vec{s}_{Q(I)} \psi_{Q(I)}$. Then $Tr(\vec{f}) = \sum_{I \in \mathcal{D}_j} \vec{t}_I b_I$, where $\vec{t}_I = \ell(I)^{-(n+1-j)/2} \vec{s}_{Q(I)}$ and $b_I(x') = \ell(I)^{(n+1-j)/2} \psi_{Q(I)}(x', 0)$.

Using the assumption (3.2),

$$\|Tr(\vec{f})\|_{\dot{B}_p^{\alpha-\frac{n+1-j}{p}+\kappa,\,q}(W)} \le c\|\vec{t}\|_{\dot{b}_p^{\alpha-\frac{n+1-j}{p}+\kappa,\,q}(W)} \le c\|\vec{s}\|_{\dot{b}_p^{\alpha q}(V)} \le c\|\vec{f}\|_{\dot{B}_p^{\alpha q}(V)},$$

similarly to the proof of Theorem 1.2.

The modifications needed for the proof of Theorem 3.2 are similar to those we have just indicated.

Finally, we remark that all of the results we have stated have analogues for the inhomogeneous matrix-weighted Besov spaces $B_p^{\alpha q}(W)$ (defined in [R1], §11, and [FR], §6). For the proofs we use Daubechies' orthonormal decomposition on $\mathbb{R}^n$ in the inhomogeneous form $f = \sum_{I\in\mathcal{D}_n,\ell(I)=1}\langle\vec{f},\gamma_I\rangle\gamma_I + \sum_{i=1}^{2^n-1}\sum_{Q\in\mathcal{D}_n,\ell(Q)<1}\langle\vec{f},\psi_Q^{(i)}\rangle\psi_Q^{(i)}$, where γ is the scaling function for Daubechies' wavelets. The only differences in the statements are that conditions (1.5) in Theorem 1.2 and (1.7) in Theorem 1.3 (also (3.2) in Theorem 3.1 and (3.3) in Theorem 3.2) are required to hold only for dyadic cubes I with $\ell(I) \le 1$, and the extension operator is defined using only terms corresponding to cubes with side lengths less than or equal to one, using γ instead of the $\psi_I^{(i)}$ for $\ell(Q) = 1$. The rest of each proof is essentially the same, using the results for the inhomogeneous spaces in [R1], §11, and [FR], §6.

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