Matrix-weighted Besov Spaces and Conditions of A_p type for $0 < p \leq 1$

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ABSTRACT. We introduce the matrix weight class A_p for $0 < p \leq 1$. For $W \in A_p$, we define the continuous and discrete matrix-weighted Besov spaces $\dot{B}_p^{\alpha q}(W)$ and $\dot{b}_p^{\alpha q}(W)$ and show their equivalence via transforms of wavelet type. We show that appropriate Calderón-Zygmund operators are bounded on $\dot{B}_p^{\alpha q}(W)$. Furthermore, we determine the duals of these Besov spaces using the technique of reducing operators.

CONTENTS

1. INTRODUCTION

Let $\mathcal{M}$ be the cone of non-negative-definite $m \times m$ complex-valued matrices. By definition, a *matrix weight* W is a locally integrable map $W : \mathbb{R}^n \to \mathcal{M}$. In the scalar case (i.e., $m = 1$), the weight will be denoted w. For a measurable vector-valued function $\vec{f} = (f_1, \dots, f_m)^T : \mathbb{R}^n \to \mathbb{C}^m$, let

$$\|\vec{f}\|_{L^p(W)} = \left(\int_{\mathbb{R}^n} \|W^{1/p}(t)\vec{f}(t)\|^p \, dt \right)^{1/p} \quad \text{for } 0 < p < \infty,$$

where $\|\cdot\|$ denotes the usual norm on $\mathbb{C}^m$.

For $1 < p < \infty$, Nazarov, Treil and Volberg (see [13], [14] or [29]) formulated the matrix A_p condition and showed that it is the right condition for Calderón-Zygmund operators to be bounded on $L^p(W)$ (e.g., for $n = 1$ it is the necessary and sufficient condition for the $L^p(W)$ boundedness of the Hilbert transform). They expressed the A_p condition in terms of dual metrics and averagings, but in [19, Corollary 3.3] it was shown that $W \in A_p$ if and only if

$$(1.1) \qquad \left(\int_Q \left(\int_Q \|W^{1/p}(t)W^{-1/p}(y)\|^p \, \frac{dt}{|Q|} \right)^{p'/p} \frac{dy}{|Q|} \right)^{p/p'} \le c,$$

for every cube $Q \subseteq \mathbb{R}^n$, where $p' = p/(p-1)$ is the usual conjugate index, and the norm inside the integral is the usual matrix operator norm. In the scalar weighted case this easily reduces to the well-known Muckenhoupt A_p condition, see [9].

In [19] and [20], the theory of matrix-weighted Besov spaces $\dot{B}_p^{\alpha q}(W)$ with $1 < p < \infty$ was developed. A satisfactory theory was shown to exist under any of the following conditions: (i) $W \in A_p$, (ii) W is a doubling matrix of order p (see the definition in Section 2) with doubling exponent $\beta_p < p$, or (iii) W is diagonal and doubling. Although these results might suggest that the theory should work under only the doubling assumption on W (and case (iii) shows that this suggestion is correct in the scalar case), an example of Nazarov, given below in Theorem 7.1, shows that the doubling condition is not sufficient in the general matrix-weighted case. For the general theory of Besov spaces in the scalar case, refer to [26] and [15]. For the discrete characterization of scalar, unweighted Besov spaces along the lines considered below, see [4], and for the related scale of Triebel-Lizorkin spaces in the scalar case, see [5], which includes the weighted case in Section 10.

Besov spaces were first developed when it was noted that they are the trace spaces of Sobolev spaces for $1 < p < \infty$ ([1]). For $p = 1$ they arise in the characterization of Fourier multiplier operators on Lipschitz spaces; more precisely, m is a bounded Fourier multiplier from Λ_α to Λ_β with $\beta > \alpha$ if and only if $\widehat{m} \in B_1^{\beta-\alpha,\infty}$ (see [31, 32] for the one-dimensional case, and [23–25] in general). An important result involving Besov spaces for $p < 1$ is Peller's theorem that a Hankel operator H_b belongs to the Schatten-Von Neumann class S_p if and only if the symbol b belongs to $B_p^{1/p,p}$, for $0 < p < \infty$ (see e.g., [16, Chapter 6]). In this paper, we extend the theory of matrix-weighted Besov spaces $\dot{B}_p^{\alpha q}(W)$ to the case $0 < p \le 1$.

Definition (Matrix A_p weight, $0 < p \le 1$)**.** For $0 < p \le 1$, we say W is a matrix A_p weight if $W : \mathbb{R}^n \to \mathcal{M}$ is a.e. invertible, W and W^{-1} are locally integrable, and

$$\|W\|_{A_p} = \sup_Q \operatorname*{ess\,sup}_{y\in Q} \frac{1}{|Q|}\int_Q \|W^{1/p}(t)W^{-1/p}(y)\|^p\,dt < \infty, \tag{1.2}$$

where the first supremum is over all cubes $Q \subseteq \mathbb{R}^n$.

Note that (1.2) is obtained by formally letting $p' \to \infty$ in (1.1). In the scalar case, condition (1.2) becomes

$$\operatorname*{ess\,sup}_{y\in Q} w^{-1}(y)\cdot\frac{1}{|Q|}\int_Q w(t)\,dt \le c \quad \text{for every } Q \subseteq \mathbb{R}^n,$$

or, equivalently,

$$\frac{1}{|Q|}\int_Q w(t)\,dt \le cw(y) \quad \text{for a.e. } y \in Q, \text{ for all } Q \subseteq \mathbb{R}^n.$$

In terms of the maximal function, this condition is

$$Mw(x) \stackrel{\text{def}}{=} \sup_{Q:x\in Q}\frac{1}{|Q|}\int_Q w(t)\,dt \le cw(x) \quad \text{for a.e. } x,$$

which is the A_1 condition for scalar weights (for example, see [22] or [8]). So in the scalar case the A_p condition (1.2) reduces to the usual A_1 condition for all $0 < p \le 1$.

We say a function $\varphi \in S(\mathbb{R}^n)$ belongs to the class $\mathcal{A}$ of admissible kernels if $\operatorname{supp}\hat{\varphi} \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\}$, and $|\hat{\varphi}(\xi)| \ge c > 0$ if $\frac{3}{5} \le |\xi| \le \frac{5}{3}$. Set $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$ for $\nu \in \mathbb{Z}$.

Definition (Matrix-weighted Besov space $\dot{B}_p^{\alpha q}(W)$)**.** For $\alpha \in \mathbb{R}$, $0 < p \le 1$, $0 < q \le \infty$, $\varphi \in \mathcal{A}$, and $W \in A_p$, the Besov space $\dot{B}_p^{\alpha q}(W)$ is the collection of all vector-valued distributions $\vec{f} = (f_1, \dots, f_m)^T$ with $f_i \in S'/\mathcal{P}(\mathbb{R}^n)$ (the space of tempered distributions modulo polynomials), $1 \le i \le m$, such that

$$\begin{aligned}\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} &= \|\{2^{\nu\alpha}\|\varphi_\nu * \vec{f}\|_{L^p(W)}\}_\nu\|_{l^q}\\ &= \Big(\sum_{\nu\in\mathbb{Z}} 2^{\nu\alpha q}\|W^{1/p}\cdot(\varphi_\nu * \vec{f})\|_{L^p}^q\Big)^{1/q} < \infty,\end{aligned}$$

where $\varphi_\nu * \vec{f} = (\varphi_\nu * f_1, \dots, \varphi_\nu * f_m)^T$, and as usual, the l^q norm is replaced by the sup on ν when $q = \infty$.

The matrix-weighted Besov spaces with $1 < p < \infty$ were introduced in [19]. As in the unweighted, scalar case ([26, p. 14]), $\|\cdot\|_{\dot{B}_p^{\alpha q}(W)}$ is a quasi-norm (norm if $p, q \geq 1$).

Let $\mathcal{D}$ be the collection of dyadic cubes in $\mathbb{R}^n$. For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $Q_{\nu k}$ denote the dyadic cube $\{(x_1, \dots, x_n) \in \mathbb{R}^n : k_i \leq 2^\nu x_i < k_i + 1,\ i = 1, \dots, n\}$ and let $x_Q = 2^{-\nu}k$ be its "lower left corner". Write $|Q|$ for the Lebesgue measure of Q and $\ell(Q)$ for the side length of Q. Let $\mathcal{D}_\nu = \{Q \in \mathcal{D} : \ell(Q) = 2^{-\nu}\}$. Set $\varphi_Q(x) = |Q|^{-1/2}\varphi(2^\nu x - k) = |Q|^{1/2}\varphi_\nu(x - x_Q)$ for $Q = Q_{\nu k}$. For each $\vec{f}$ with $f_i \in S'(\mathbb{R}^n)/\mathcal{P}$ we define the *φ-transform* S_φ to be the map taking $\vec{f}$ to the vector-valued sequence $S_\varphi(\vec{f}\,) = \{\langle \vec{f}, \varphi_Q \rangle\}_Q = \{(\langle f_1, \varphi_Q \rangle, \dots, \langle f_m, \varphi_Q \rangle)^T\}_Q$ for Q dyadic. We call $\vec{s}_Q(\vec{f}\,) := \langle \vec{f}, \varphi_Q \rangle$ the *φ-transform* coefficient of $\vec{f}$.

For each $\varphi \in \mathcal{A}$ there exists $\psi \in \mathcal{A}$ (see e.g. [6]) such that

$$\sum_{\nu \in \mathbb{Z}} \overline{\hat{\varphi}(2^\nu \xi)} \cdot \hat{\psi}(2^\nu \xi) = 1, \quad \text{if } \xi \neq 0. \tag{1.3}$$

We say that (φ, ψ) is a pair of mutually admissible kernels if φ, $\psi \in \mathcal{A}$, and (1.3) holds.

Similar to φ_Q, define $\psi_Q(x) = |Q|^{-1/2}\psi(2^\nu x - k)$ for $Q = Q_{\nu k}$. The *inverse φ-transform* T_ψ is the map taking a sequence $s = \{s_Q\}_Q$ to the distribution $T_\psi s = \sum_Q s_Q \psi_Q$. In the vector case, we set $\vec{s} = \{\vec{s}_Q\}_{Q \in \mathcal{D}}$, where $\vec{s}_Q = (s_Q^{(1)}, \dots, s_Q^{(m)})^T$. Then we let $T_\psi \vec{s} = \sum_Q \vec{s}_Q \psi_Q$, where

$$\vec{s}_Q \psi_Q = (s_Q^{(1)} \psi_Q, \dots, s_Q^{(m)} \psi_Q)^T.$$

The φ-transform decomposition (see [5]) states that for all $\vec{f}$ with each component $f_j \in S'/\mathcal{P}$,

$$\vec{f} = \sum_{Q \in \mathcal{D}} \langle \vec{f}, \varphi_Q \rangle \psi_Q. \tag{1.4}$$

In other words, $T_\psi \circ S_\varphi$ is the identity.

In the classical case, there are many equivalent definitions of Besov spaces in the literature (for an overview, see [15]). For example, one can use an integral over $t > 0$ instead of a sum over $\nu \in \mathbb{Z}$ and one can replace the kernels $\{\varphi_\nu\}_{\nu \in \mathbb{Z}}$ with dilations of other kernels, including appropriate derivatives of the Poisson kernel (see [23–25] or [21, Chapter 5]). We will not attempt to establish all such analogous equivalences for the matrix-weighted case. We make a couple of remarks, however. We chose kernels with compactly supported Fourier transforms in order to obtain (1.4). Also, the numbers $\frac{1}{2}$, 2, $\frac{3}{5}$, and $\frac{5}{3}$ in the definition of the class $\mathcal{A}$ are traditional and convenient but not essential. For example, one can replace $\frac{1}{2}$ by a, $\frac{3}{5}$ by b, $\frac{5}{3}$ by c, and 2 by $4a$, as long as $2b < c$; and there are variations where $\frac{1}{2}$ and 2 are replaced by A and $1/A$. What is important is that the supports of $\hat{\varphi}_\nu$ overlap in a bounded fashion; more precisely, one needs $0 < c_1 \le \sum_{\nu\in\mathbb{Z}} |\hat{\varphi}_\nu(\xi)|^2 \le c_2 < \infty$ for all $\xi \ne 0$, which allows one to obtain an appropriate version of (1.3). Defining a quasi-norm as in Definition 1 using kernels satisfying these conditions yields an equivalent quasi-norm; this is proved by a variant of the argument ([19, Theorem 1.8]) referred to below which shows that $\dot{B}_p^{\alpha q}(W)$ does not depend on the choice of $\varphi \in \mathcal{A}$.

Definition (Matrix-weighted sequence Besov space $\dot{b}_p^{\alpha q}(W)$)**.** For $\alpha \in \mathbb{R}$, $0 < p \le 1$, $0 < q \le \infty$, and W a matrix weight, the space $\dot{b}_p^{\alpha q}(W)$ consists of all vector-valued sequences $\vec{s} = \{\vec{s}_Q\}_{Q\in\mathcal{D}}$ such that

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} = \Big\|\Big\{2^{\nu\alpha}\Big\|\sum_{Q\in\mathcal{D}_\nu} |Q|^{-1/2}\vec{s}_Q\chi_Q\Big\|_{L^p(W)}\Big\}_\nu\Big\|_{l^q}$$
$$= \Big(\sum_{\nu\in\mathbb{Z}}\Big\|2^{\nu\alpha}\sum_{Q\in\mathcal{D}_\nu}|Q|^{-1/2}\|W^{1/p}(t)\vec{s}_Q\|\chi_Q(t)\Big\|^q_{L^p(dt)}\Big)^{1/q} < \infty.$$

The main conclusions about the spaces $\dot{B}_p^{\alpha q}(W)$ will be derived from the following equivalence.

Theorem 1.1. *Let $\alpha \in \mathbb{R}$, $0 < q \le \infty$, $0 < p \le 1$, and $W \in A_p$. Then*

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} = \Big\|\sum_{Q\in\mathcal{D}} \vec{s}_Q(\vec{f})\psi_Q\Big\|_{\dot{B}_p^{\alpha q}(W)} \approx \|\{\vec{s}_Q(\vec{f})\}_Q\|_{\dot{b}_p^{\alpha q}(W)}, \tag{1.5}$$

where $\{\vec{s}_Q(\vec{f})\}_Q$ is the sequence of φ-transform coefficients of $\vec{f}$.

A similar equivalence holds for appropriate wavelet coefficients, see Theorem 4.4. We remark that one direction of (1.5), namely the estimate $\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} \le C\|\{\vec{s}_Q(\vec{f})\}_Q\|_{\dot{b}_p^{\alpha q}(W)}$, holds in a more general form (see Theorem 3.1) and requires only the doubling property for W (see Corollary 3.2). In the case of a diagonal matrix weight W (equivalently, in the scalar case), the doubling condition alone is sufficient to obtain (1.5). We don't know if the A_p condition in Theorem 1.1 can be relaxed in general.

Using Theorem 1.1, the fact that the space $\dot{B}_p^{\alpha q}(W)$ is independent of the choice of admissible φ, in the sense that two admissible φ's yield equivalent norms, follows by the same argument as for $p > 1$ in [19, Section 6 and Theorem 1.8]. Also, one obtains that the spaces $\dot{B}_p^{\alpha q}(W)$ are complete, as in [19, Corollary 7.2]. Hence $\dot{B}_p^{\alpha q}(W)$ is a quasi-Banach space (Banach if $p, q \ge 1$).

In Section 4, we prove that for $W \in A_p$, certain classes of Calderón-Zygmund operators are bounded on $\dot{B}_p^{\alpha q}(W)$, $0 < p \le 1$. In Section 5 we calculate the dual spaces of $\dot{B}_p^{\alpha q}(W)$ and $\dot{b}_p^{\alpha q}(W)$ for $0 < p \le 1$. For this, we need to consider a sequence of "reducing operators" $\{A_Q\}_{Q\in\mathcal{D}}$ of order p for W. We define associated reducing operator Besov spaces $\dot{b}_p^{\alpha q}(\{A_Q\})$ and $\dot{B}_p^{\alpha q}(\{A_Q\})$. Reducing operators provide another level of discretization that allows us to obtain weighted duality results from the unweighted case. In order to express this duality in terms of the weight W, we introduce the "uncorrelated" Besov spaces $\dot{b}_{pr}^{\alpha q}(W)$ and $\dot{B}_{pr}^{\alpha q}(W)$. In terms of these spaces, our main duality results state that

$$[\dot{b}_p^{\alpha q}(W)]^* \approx \dot{b}_\infty^{-\alpha+n(1/p-1),q'}(\{A_Q^{-1}\}) \approx \dot{b}_{\infty p}^{-\alpha+n(1/p-1),q'}(W^{-1}), \tag{1.6}$$

(see Theorems (5.3) and (5.6)), and

$$[\dot{B}_p^{\alpha q}(W)]^* \approx \dot{B}_\infty^{-\alpha+n(1/p-1),q'}(\{A_Q^{-1}\}) \approx \dot{B}_{\infty p}^{-\alpha+n(1/p-1),q'}(W^{-1}), \tag{1.7}$$

(see Theorems (5.9) and (5.12)).

Most of these results (except those about boundedness of Calderón-Zygmund operators in the case $p \le 1$) have analogues for the inhomogeneous matrix-weighted Besov spaces $b_p^{\alpha q}(W)$ and $B_p^{\alpha q}(W)$, via slight modifications of the proofs. For reference, these results are stated in Section 6.

The example of Nazarov noted above is presented in Section 7. It extends to the case $p \le 1$ to show that the doubling condition alone is not sufficient to obtain (1.5).

2. DOUBLING MATRIX WEIGHTS

For a cube $Q \subseteq \mathbb{R}^n$, let $2Q$ denote the cube with the same center and orientation but with twice the side length.

Definition (Doubling matrix weight)**.** A (non-zero) matrix weight W is called a *doubling matrix weight of order* p, $0 < p < \infty$, if there exists a constant c such that for all cubes $Q \subseteq \mathbb{R}^n$ and all $y \in \mathbb{C}^m$,

$$\int_{2Q} \|W^{1/p}(t)y\|^p \, dt \le c \int_Q \|W^{1/p}(t)y\|^p \, dt, \tag{2.1}$$

i.e., the scalar measures $w_y(t) = \|W^{1/p}(t)y\|^p$, for $y \in \mathbb{C}^m$, are uniformly doubling. If $c = 2^\beta$ is the smallest constant for which (2.1) holds, then β is called the *doubling exponent* of W.

It is well known (see e.g., [19]) that the doubling property $\mu(2Q)) \le 2^\beta \mu(Q)$ (for any cube $Q \subseteq \mathbb{R}^n$) implies

$$\frac{\mu(F)}{\mu(E)} \le c\left(\frac{|F|}{|E|}\right)^{\beta/n}, \tag{2.2}$$

where F is a ball (or a cube) and $E \subseteq F$ is a sub-ball (sub-cube) (not *any* subset of F; any subset would be equivalent to A_∞ condition, see [22]). In particular,

$$\mu(Q_{\nu k}) \le c(1 + |k - m|)^\beta \mu(Q_{\nu m}). \tag{2.3}$$

Lemma 2.1. *Let $0 < p \le 1$. If $W \in A_p$, then for any $x \in \mathbb{C}^m$,*

(a) *$w_x(t) = \|W^{1/p}(t)x\|^p$ is a scalar A_1 weight: in fact, $Mw_x(t) \le \|W\|_{A_p} w_x(t)$ for a.e. t;*

(b) *W is a doubling matrix of order p, with doubling constant determined only by $\|W\|_{A_p}$.*

Proof. Let $Q \subseteq \mathbb{R}^n$ be a cube. We need to show that

$$\frac{1}{|Q|}\int_Q \|W^{1/p}(s)x\|^p\,ds \le \|W\|_{A_p}\,\|W^{1/p}(t)x\|^p, \tag{2.4}$$

for a.e. $t \in Q$. Let $y = W^{1/p}(t)x$. Then the left hand side of (2.4) is

$$\begin{aligned}&\frac{1}{|Q|}\int_Q \|W^{1/p}(s)W^{-1/p}(t)y\|^p\,ds\\ &\qquad \le \frac{1}{|Q|}\int_Q \|W^{1/p}(s)W^{-1/p}(t)\|^p\,ds\,\|y\|^p \le \|W\|_{A_p}\,\|y\|^p,\end{aligned}$$

for a.e. $t \in Q$, as desired.

For part (b), the doubling property for W is just the property that w_x is a scalar doubling measure (uniformly in x), which holds because we have just seen that w_x is a scalar A_1 weight (uniformly in x). ∎

For $0 < p \le 1$, we in fact have that $W \in A_p$ if and only if the scalar weights $w_x(t)$ in Lemma 2.1 are uniformly in A_1. To see this, suppose (2.4) holds with $\|W\|_{A_p}$ replaced by a constant, for a.e. $t \in Q$. Then substituting $y = W^{1/p}(t)x$ shows that $W \in A_p$.

3. Norm Equivalence Between Continuous and Discrete Besov Spaces

Theorem 1.1, the norm equivalence between $\dot{B}_p^{\alpha q}(W)$ and $\dot{b}_p^{\alpha q}(W)$, will follow from the boundedness of the inverse φ-transform (Corollary 3.2) and the boundedness of the φ-transform (Theorem 3.4).

Let W be a doubling matrix weight of order p with doubling exponent β. For $0 < \delta \le 1$, $M > 0$ and $N \in \mathbb{Z}$ define m_Q to be a *smooth* (δ, M, N)*-molecule* for Q dyadic (see Section 5 of [19]) if:

(M1) $\int x^\gamma m_Q(x)\,dx = 0$, for $|\gamma| \le N$,

(M2) $|m_Q(x)| \le |Q|^{-1/2}(1 + |x - x_Q|/\ell(Q))^{-\max(M, M-\alpha)}$,

(M3) $|D^\gamma m_Q(x)| \le |Q|^{-1/2-|\gamma|/n}(1 + |x - x_Q|/\ell(Q))^{-M}$ if $|\gamma| \le [\alpha]$,

(M4) $|D^\gamma m_Q(x) - D^\gamma m_Q(y)| \le |Q|^{-1/2-|\gamma|/n-\delta/n}|x - y|^\delta$ $\sup_{|z|\le|x-y|}(1 + |x - z - x_Q|/\ell(Q))^{-M}$ if $|\gamma| = [\alpha]$.

It is understood that (M1) is void if $N < 0$; and (M3), (M4) are void if $\alpha < 0$. Also, $[\alpha]$ stands for the greatest integer $\le \alpha$; γ is a multi-index $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_i \in \mathbb{N} \cup \{0\}$, $1 \le i \le n$, and the standard multi-index notation is used.

We say $\{m_Q\}_Q$ is a *family* of *smooth molecules* for $\dot{B}_p^{\alpha q}(W)$, $0 < p \le 1$, $\alpha \in \mathbb{R}$ and $0 < q \le \infty$, if each m_Q is a (δ, M, N)-molecule as above, with

(M.i) $\alpha - [\alpha] < \delta \le 1$,
(M.ii) $M > \beta/p$,
(M.iii) $N = \max([\beta/p - n - \alpha], -1)$.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$, $0 < p \le 1$, $0 < q \le \infty$, and let W be a doubling matrix weight of order p. Suppose $\{m_Q\}_Q$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$. Then*

$$\Big\| \sum_{Q \in \mathcal{D}} \vec{s}_Q m_Q \Big\|_{\dot{B}_p^{\alpha q}(W)} \le c \| \{\vec{s}_Q\}_Q \|_{\dot{b}_p^{\alpha q}(W)}. \tag{3.1}$$

The proof is essentially the same as for Theorem 5.2 in [19] with the triangle inequality for the L^p norm replaced by the quasi-norm property $\|f + g\|_{L^p} \le c_p(\|f\|_{L^p} + \|g\|_{L^p})$, and the conjugate exponent p' regarded as ∞ for $p \le 1$ (and so $n/p' \equiv 0$).

Corollary 3.2. *Let $\alpha \in \mathbb{R}$, $0 < p \le 1$, $0 < q \le \infty$, and let W be a doubling matrix weight of order p. For each Q dyadic, let $\vec{s}_Q := \langle \vec{f}, \varphi_Q \rangle$ be the φ-transform coefficient of $\vec{f}$. Then*

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} \le C \| \{\vec{s}_Q\}_Q \|_{\dot{b}_p^{\alpha q}(W)}, \tag{3.2}$$

where C depends only on α, q, p, and the doubling constant of W.

Proof. Let ψ be as in (1.3). It is easy to see that $\{\psi_Q\}_Q$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$, for all $0 < p \le 1$, $\alpha \in \mathbb{R}$ and $0 < q \le \infty$. Hence, the result follows from (1.4) and Theorem 3.1. □

Definition. For $\nu \in \mathbb{Z}$, let $E_\nu = \{\vec{f} : f_i \in S'$, and $\operatorname{supp} \hat{f}_i \subseteq \{\xi \in \mathbb{R}^n : |\xi| \le 2^{\nu+1}\}$, $i = 1, \dots, m\}$. We say E_ν consists of vector-functions of exponential type $2^{\nu+1}$.

Denote $\Gamma = \{\gamma \in S : \hat{\gamma} = 1$ if $|\xi| \le 2$ and $\operatorname{supp} \hat{\gamma} \subseteq \{\xi \in \mathbb{R}^n : |\xi| < \pi\}\}$. Before we establish the boundedness of the φ-transform, we prove a weighted sampling estimate for exponential type functions.

Lemma 3.3. *Let $0 < p \le 1$, $W \in A_p$, and $\vec{g} \in E_0$. Then*

$$\sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(t)\vec{g}(k)\|^p \, dt \le c_{p,W} \|\vec{g}\|_{L^p(W)}^p. \tag{3.3}$$

Proof. Selecting $\gamma \in \Gamma$, we get the identity $\vec{g}(x) = \sum_{\ell \in \mathbb{Z}^n} \vec{g}(\ell)\gamma(x - \ell)$ (see [6, p.55]). Applying this to $\vec{g}^{\,y} \in E_0$, which is defined by $\vec{g}^{\,y}(x) = \vec{g}(x + y)$, we get $\vec{g}(x+y) = \sum_{\ell\in\mathbb{Z}^n} \vec{g}(\ell+y)\gamma(x-\ell)$. Letting $x = k-y$ gives

$$\vec{g}(k) = \sum_{\ell\in\mathbb{Z}^n} \vec{g}(\ell+y)\gamma(k-y-\ell). \tag{3.4}$$

Hence, for all $y \in Q_{00}$, we have

$$\begin{aligned}\|W^{1/p}(t)\vec{g}(k)\|^p &= \Big\|\sum_{\ell\in\mathbb{Z}^n} W^{1/p}(t)\vec{g}(\ell+y)\gamma(k-y-\ell)\Big\|^p \\ &\le \Big(\sum_{\ell\in\mathbb{Z}^n} \|W^{1/p}(t)\vec{g}(\ell+y)\|\,|\gamma(k-y-\ell)|\Big)^p \\ &\le c_M \sum_{\ell\in\mathbb{Z}^n} \frac{\|W^{1/p}(t)\vec{g}(\ell+y)\|^p}{(1+|k-\ell|)^M},\end{aligned}$$

where we can choose M as large as we like. We average this estimate over $y \in Q_{00}$ to obtain

$$\begin{aligned}&\|W^{1/p}(t)\vec{g}(k)\|^p \\ &\quad\le c_M \sum_{\ell\in\mathbb{Z}^n} \frac{1}{(1+|k-\ell|)^M} \int_{Q_{00}} \|W^{1/p}(t)\vec{g}(\ell+y)\|^p\,dy \\ &\quad= c_M \sum_{\ell\in\mathbb{Z}^n} \frac{1}{(1+|k-\ell|)^M} \int_{Q_{0\ell}} \|W^{1/p}(t)\vec{g}(y)\|^p\,dy.\end{aligned} \tag{3.5}$$

Integrating this estimate over Q_{0k}, interchanging the order of integration, and applying (2.3) to the doubling measure $w_y(t) = \|W^{1/p}(t)\vec{g}(y)\|^p$ (Lemma 2.1), we get

$$\begin{aligned}&\int_{Q_{0k}} \|W^{1/p}(t)\vec{g}(k)\|^p\,dt \\ &\quad\le c_M \sum_{\ell\in\mathbb{Z}^n} \frac{1}{(1+|k-\ell|)^M} \int_{Q_{0\ell}}\int_{Q_{0k}} \|W^{1/p}(t)\vec{g}(y)\|^p\,dt\,dy \\ &\quad\le c_{M,\beta} \sum_{\ell\in\mathbb{Z}^n} \frac{(1+|k-\ell|)^\beta}{(1+|k-\ell|)^M} \int_{Q_{0\ell}}\int_{Q_{0\ell}} \|W^{1/p}(t)\vec{g}(y)\|^p\,dt\,dy.\end{aligned}$$

Summing over $k \in \mathbb{Z}^n$, and taking $M > n+\beta$ so that

$$\sum_{k\in\mathbb{Z}^n} (1+|k-\ell|)^{-M+\beta} < c_{M,\beta,n},$$

we get

$$\sum_{k\in\mathbb{Z}^n}\int_{Q_{0k}}\|W^{1/p}(t)\vec{g}(k)\|^p\,dt \le c_{M,\beta,n}\sum_{\ell\in\mathbb{Z}^n}\int_{Q_{0\ell}}\int_{Q_{0\ell}}\|W^{1/p}(t)\vec{g}(y)\|^p\,dt\,dy. \tag{3.6}$$

Writing $\|W^{1/p}(t)\vec{g}(y)\|^p \le \|W^{1/p}(t)W^{-1/p}(y)\|^p\,\|W^{1/p}(y)\vec{g}(y)\|^p$, we obtain

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n}\int_{Q_{0k}}\|W^{1/p}(t)\vec{g}(k)\|^p\,dt\\
&\quad\le c_{M,\beta,n}\sum_{\ell\in\mathbb{Z}^n}\int_{Q_{0\ell}}\|W^{1/p}(y)\vec{g}(y)\|^p\int_{Q_{0\ell}}\|W^{1/p}(t)W^{-1/p}(y)\|^p\,dt\,dy\\
&\quad\le c_{M,\beta,n}\|W\|_{A_p}\sum_{\ell\in\mathbb{Z}^n}\int_{Q_{0\ell}}\|W^{1/p}(y)\vec{g}(y)\|^p\,dy\\
&\quad= c_{M,\beta,n}\|W\|_{A_p}\|\vec{g}\|^p_{L^p(W)}.
\end{aligned}$$

□

Now we can show the boundedness of the φ-transform.

Theorem 3.4. *Let $\alpha\in\mathbb{R}$, $0<q\le\infty$, $0<p\le1$, and $W\in A_p$. Then there exists c such that for all $\vec{f}\in\dot{B}^{\alpha q}_p(W)$,*

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}^{\alpha q}_p(W)} \le c\|\vec{f}\|_{\dot{B}^{\alpha q}_p(W)}, \tag{3.7}$$

where $\vec{s}_Q=(S_\varphi\vec{f})_Q=\langle\vec{f},\varphi_Q\rangle$.

The proof follows the same steps as in Theorem 6.6 of [19] with Lemmas 6.3 and 6.5 being replaced by Lemma 3.3: a dilation argument yields the analogue of (3.3) for $\vec{g}=\varphi_\nu*\vec{f}\in E_\nu$; then multiplying by $2^{\nu\alpha}$ and taking the ℓ^q norm over $\nu\in\mathbb{Z}$, yields (3.7).

The argument in [19, Lemma 6.5] carries over to the case $0<p\le1$ to show that the conclusions of Theorem 1.1 hold in the scalar case ($m=1$), or equivalently, the case where $W(t)$ is diagonal for a.e. t, under just the doubling condition.

4. OPERATORS ON BESOV SPACES FOR $0<p\le1$

In this section we consider almost diagonal matrices on $\dot{b}^{\alpha q}_p(W)$, almost diagonal operators, Calderón-Zygmund operators, and wavelet transforms on $\dot{B}^{\alpha q}_p(W)$. Having established the norm equivalence between $\dot{B}^{\alpha q}_p(W)$ and $\dot{b}^{\alpha q}_p(W)$ for $p\le1$, we find that the theory of the above mentioned operators on Besov spaces is similar to the case $p>1$ studied in Section 8, 9 and 10 of [19]. Below we indicate the necessary changes.

Definition. A matrix $A = (a_{QP})_{Q,P}$ is *almost diagonal* of order (α_1, α_2, M), $A \in \mathbf{ad}(\alpha_1, \alpha_2, M)$, if there exists $c > 0$ such that for all $Q, P \in \mathcal{D}$,

$$|a_{QP}| \leq c \min\left(\left[\frac{\ell(Q)}{\ell(P)}\right]^{\alpha_1}, \left[\frac{\ell(P)}{\ell(Q)}\right]^{\alpha_2}\right)\left(1 + \frac{|x_Q - x_P|}{\max(\ell(Q), \ell(P))}\right)^{-M}. \tag{4.1}$$

We define the action of $A = (a_{QP})_{Q,P}$ on a sequence $\vec{s} = \{\vec{s}_Q\}_Q$ componentwise: $A\vec{s} = \{(A\vec{s})_Q\}_Q$, where the components of $(A\vec{s})_Q$ are $(A\vec{s})_Q^{(i)} = \sum_P a_{QP} s_P^{(i)}$, for $i = 1, 2, \ldots, m$.

Theorem 4.1. *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p \leq 1$ and let W be a doubling matrix of order p with doubling exponent β. Suppose $A \in \mathbf{ad}(\alpha_1, \alpha_2, M)$ with $\alpha_1 > \alpha + n/2$, $\alpha_2 > \beta/p - \alpha - n/2$, and $M > \beta/p$. Then $A : \dot{b}_p^{\alpha q}(W) \longrightarrow \dot{b}_p^{\alpha q}(W)$ is bounded.*

The proof of this theorem is easier than its analogue for $p > 1$ (see p. 296-298 in [19]), since one of the difficulties of bringing the p^{th} power inside the infinite sum, for which the decay of a_{QP} and Hölder's inequality were used, is trivial for $0 < p \leq 1$: $(\sum_i a_i)^p \leq \sum_i a_i^p$.

Definition. Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p \leq 1$, and $\beta \geq n$. Let T be a continuous linear operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. Then T is an *almost diagonal operator*, $T \in \mathbf{AD}_p^{\alpha q}(\beta)$, if for some pair of mutually admissible kernels (φ, ψ) the matrix $(a_{QP}) \in \mathbf{ad}(\alpha_1, \alpha_2, M)$, with $\alpha_1 > \alpha + n/2$, $\alpha_2 > \beta/p - \alpha - n/2$, and $M > \beta/p$, where $a_{QP} = \langle T\psi_P, \varphi_Q\rangle$.

We let such an operator T act on $\vec{f}$ componentwise: $T(\vec{f}) = (Tf_1, Tf_2, \ldots, Tf_m)^T$. The facts that $T \in \mathbf{AD}_p^{\alpha q}(\beta)$ is independent of the choice of the pair (φ, ψ) and that it extends to a bounded operator on $\dot{B}_p^{\alpha q}(W)$ if $W \in A_p$ with doubling exponent β, hold in the same manner as in [19] with the proper replacements. (For example, note that $\mathcal{S}_0$ is dense in $\dot{B}_p^{\alpha q}(W)$ if $q < \infty$, $p > 0$ and $W \in A_p$, which follows from the norm equivalence (1.5). For more details see the Appendix in [18].)

For $N > 0$ and $0 < \varepsilon \leq 1$, let $CZO(N + \varepsilon)$ denote the class of integral operators $(Tf)(x) = \int K(x, y) f(y)\, dy$ which are continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, with kernel K satisfying

(I) $|K(x, y)| \leq c/|x - y|^n$,

(II_N) $|D_{(2)}^{\gamma} K(x, y)| \leq c/|x - y|^{n+|\gamma|}$ for $|\gamma| \leq N$,

($\mathrm{II}_{N+\varepsilon}$) $|D_{(2)}^{\gamma} K(x, y) - D_{(2)}^{\gamma} K(x', y)| + |D_{(2)}^{\gamma} K(y, x) - D_{(2)}^{\gamma} K(y, x')| \leq c(|x - x'|^{\varepsilon}/|x - y|^{n+|\gamma|+\varepsilon})$ for $2|x - x'| \leq |x - y|$ and $|\gamma| = N$.

If $T \in CZO(N + \varepsilon)$ and its kernel $K(x, y) = K(x - y)$ also satisfies

$$\int_{R_1 < |x| < R_2} K(x)\, dx = 0, \quad \text{for all } 0 < R_1 < R_2 < \infty, \tag{C}$$

then we say that T is a convolution CZO.

With necessary replacements for $0 < p \le 1$ case, we obtain the following statement.

Theorem 4.2. *Let $0 \le \alpha < 1$, $0 < q < \infty$, $0 < p \le 1$, and $W \in A_p$. Suppose $\alpha < \varepsilon < 1$, $N + \varepsilon > \beta/p - n$, and $T \in CZO(N + \varepsilon)$. Then T extends to be a bounded operator on $\dot{B}_p^{\alpha q}(W)$, provided either of the following holds:*

(1) *$T \in WBP$, $T1 = 0$ and $T^*(y^\gamma) = 0$ for $|\gamma| \le \max([\beta/p - n - \alpha], -1)$; or*
(2) *T is a convolution CZO.*

For the proof, just note that Theorems 9.8 and 9.14 in [19] show that T maps appropriate atoms into multiples of smooth (δ, M, N)-molecules with δ, M and N satisfying (M.i)-(M.iii) above.

Remark 4.3. Part (2) of Theorem 4.2 shows that the Hilbert and Riesz transforms are bounded on $\dot{B}_p^{\alpha q}(W)$ if $W \in A_p$, not only for $1 < p < \infty$ but also for $0 < p \le 1$.

We conclude this section with the norm equivalence for wavelet transforms, i.e., if the φ-transform coefficients are replaced with certain wavelet coefficients.

The following theorem is an extension of Theorem 10.2 and Corollary 10.3 in [19] to the case $0 < p \le 1$. The proof is similar with the new norm equivalence (1.5) and other replacements such as $n/p' = 0$. In its statement, $\{\psi^{(i)}\}$, $i = 1, \dots, 2^n - 1$ denotes either Meyer's wavelets on $\mathbb{R}^n$ (see [11] or [12]) or Daubechies D_N compactly supported wavelets (see [3]), for sufficiently large N (the size required for N is determined by the indices α, q, and p, in order for Daubechies' wavelets to have the necessary smoothness and number of vanishing moments).

Theorem 4.4. *Let $0 < p \le 1$ and suppose $W \in A_p$. Also let $\alpha \in \mathbb{R}$ and $0 < q \le \infty$. Then*

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} \approx \sum_{i=1}^{2^n-1} \|\{\langle \vec{f}, \psi_Q^{(i)} \rangle\}_Q\|_{\dot{b}_p^{\alpha q}(W)}.$$

5. DUALITY

In order to characterize dual spaces of matrix-weighted Besov spaces, we use the technique of reducing operators, and then introduce a new notation in order to express duals directly in terms of the matrix weight. When we say $X^* \approx Y$ with the pairing $\langle x, y \rangle = \dots$, we mean that for every $y \in Y$, the map $x \to \langle x, y \rangle$ is a bounded linear functional on X with norm equivalent to $\|y\|_Y$, and every bounded linear functional on X is of this form for some $y \in Y$.

Reducing operators. Given a matrix weight W, we define

$$\rho_{p,Q}(x) = \left(\frac{1}{|Q|} \int_Q \|W^{1/p}(t)x\|^p \, dt \right)^{1/p},$$

for all dyadic cubes Q. For $1 \le p < \infty$, $\rho_{p,Q}$ is a norm on $\mathbb{C}^m$. For $0 < p \le 1$, $\rho_{p,Q}$ is a quasi-norm, which also satisfies that the p^{th} power is sub-additive:

$$(\rho_{p,Q}(x+y))^p \le (\rho_{p,Q}(x))^p + (\rho_{p,Q}(y))^p. \tag{5.1}$$

We say that $\{A_Q\}_{Q\in\mathcal{D}}$ is a sequence of reducing operators of order p for W if each A_Q is a positive definite matrix satisfying $\rho_{p,Q}(x) \approx \|A_Q(x)\|$, with equivalence constants independent of $x \in \mathbb{C}^m$ and $Q \in \mathcal{D}$. As observed in [29], for $1 \le p < \infty$, any matrix weight has a sequence of reducing operators of order p, by a theorem of Fritz John ([10], or see [17, Chapter 3]) that any norm in a finite-dimensional space is equivalent to a norm of the form $\rho(x) = \|Ax\|$, with equivalence constants depending only on the dimension.

For the case $0 < p < 1$, it takes a little more to prove the existence of reducing operator sequences. Given W and hence $\rho_{p,Q}$, let $C = \{x \in \mathbb{C}^m : \rho_{p,Q}(x) < 1\}$ be the unit ball associated to $\rho_{p,Q}$. Since $\rho_{p,Q}$ is a quasi-norm, C may not be convex. Let D be the convex hull of C. The key point is that $D \subseteq (2m+1)^{1/p-1}C$. Assuming this for the moment, let $q(x) = \inf\{t > 0 : x/t \in D\}$ be the norm defined by the Minkowski functional of D. Then the unit ball of q is D, and the containments $C \subseteq D \subseteq (2m+1)^{1/p-1}C$ guarantee that $\rho_{p,Q}$ and q are equivalent, with constants depending only on p and the dimension m. Applying John's theorem to q yields the existence of the desired reducing operator A_Q. To prove the asserted containment, let $x \in D$. Since $D \subset \mathbb{C}^m \approx \mathbb{R}^{2m}$, a classical theorem of Carathéodory ([2], or see e.g., [30, p. 55]) states that x can be written as a convex combination of no more than $2m+1$ points of C, say $x = \sum_{j=1}^{2m+1} \alpha_j x_j$. Then by (5.1),

$$(\rho_{p,B}(x))^p \le \sum_{j=1}^{2m+1} (\rho_{p,B}(\alpha_j x_j))^p = \sum_{j=1}^{2m+1} \alpha_j^p (\rho_{p,B}(x_j))^p < \sum_{j=1}^{2m+1} \alpha_j^p,$$

since each x_j belongs to the unit ball of $\rho_{p,B}$. Applying Hölder's inequality with exponents $s = 1/p$ and $s' = 1/(1-p)$, we obtain

$$(\rho_{p,B}(x))^p \le \sum_{j=1}^{2m+1} \alpha_j \Big(\sum_{j=1}^{2m+1} 1 \Big)^{1-p} = (2m+1)^{1-p}.$$

Any matrix-weighted space has a correspondent "reducing operator" space, which is easier to study, since instead of dealing with matrix weights, we consider a sequence of matrices constant on dyadic cubes, and establish properties of Besov spaces with such sequences. Using the correspondence between the matrix-weighted space and the reducing operator space, we obtain desired results for the initial spaces. Denote by $\mathcal{RS}_\mathcal{D}$ (reducing sequences) the collection of all sequences $\{A_Q\}_{Q\in\mathcal{D}}$ of nonnegative-definite $m \times m$ complex valued matrices.

Definition (Averaging matrix-weighted Besov space $\dot{b}_p^{\alpha q}(\{A_Q\})$). Let $\{A_Q\}_Q \in \mathcal{RS}_\mathcal{D}$ and $\alpha \in \mathbb{R}$, $0 < q \le \infty$, $0 < p \le \infty$. Define

$$\dot{b}_p^{\alpha q}(\{A_Q\}) = \Big\{\vec{s} = \{\vec{s}_Q\}_Q : \\ \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} = \Big\|\Big\{2^{\nu\alpha}\Big\|\sum_{Q\in\mathcal{D}_\nu} |Q|^{-1/2}\,\|A_Q\vec{s}_Q\|\chi_Q\Big\|_{L^p}\Big\}_\nu\Big\|_{l^q} < \infty\Big\}.$$

For $p > 1$, $\dot{b}_p^{\alpha q}(\{A_Q\})$ was defined and studied in [20]. Observe that

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} = \|\{A_Q\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\mathbb{C}^m)}, \tag{5.2}$$

where $\dot{b}_p^{\alpha q}(\mathbb{C}^m)$ is the unweighted sequence Besov space, i.e., $\dot{b}_p^{\alpha q}(W)$ with $W = I$. The correspondence between sequence Besov spaces is as follows.

Lemma 5.1. *If W is a matrix weight and $\{A_Q\}_Q$ is a sequence of reducing operators of order p for W, then for $\alpha \in \mathbb{R}$, $0 < q \le \infty$, and $0 < p < \infty$, we have $\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \approx \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}$.*

The proof is simple and follows as in [19, Lemma 7.1], for the case $1 < p < \infty$.

Duality of discrete Besov spaces. Using reducing operators we can find the dual of $\dot{b}_p^{\alpha q}(W)$. We begin with the unweighted case. The continuous analogue of the following result is standard, but the elementary proof in the discrete case is not in the literature, as far as we know.

Lemma 5.2. *Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $0 < p \le 1$. Then*

$$[\dot{b}_p^{\alpha q}(\mathbb{C}^m)]^* \approx \dot{b}_\infty^{-\alpha+n(1/p-1),q'}(\mathbb{C}^m), \tag{5.3}$$

with the pairing $\langle \vec{s}, \vec{t}\,\rangle = \sum_{Q\in\mathcal{D}} \langle \vec{s}_Q, \vec{t}_Q\rangle$.

Proof. It suffices to consider the case $m = 1$. Let $\gamma = -\alpha + n(1/p - 1)$. For the embedding $\dot{b}_\infty^{\gamma q'} \subseteq [\dot{b}_p^{\alpha q}]^*$, observe that for $s = \{s_Q\}_Q$ and $t = \{t_Q\}_Q$ we have

$$\begin{aligned}
|\langle s, t\rangle| &\le \sum_{\nu\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_\nu} |s_Q|\,|t_Q| \\
&= \sum_{\nu\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_\nu} |Q|^{-1/2+1/p}\,|s_Q|\,|Q|^{-1/p+1/2}|t_Q| \\
&\le \sum_{\nu\in\mathbb{Z}} \Big(\sum_{Q\in\mathcal{D}_\nu} \big(|Q|^{-1/2+1/p}\,|s_Q|\big)^p\Big)^{1/p} 2^{\nu n(1/p-1)} \sup_{Q\in\mathcal{D}_\nu} (|Q|^{-1/2}\,|t_Q|) \\
&\le \sum_{\nu\in\mathbb{Z}} 2^{\nu\alpha}\Big\|\sum_{Q\in\mathcal{D}_\nu} |Q|^{-1/2} s_Q\chi_Q\Big\|_{L^p}\Big\|2^{\nu(-\alpha+n(1/p-1))}\sum_{Q\in\mathcal{D}_\nu} |Q|^{-1/2} t_Q\chi_Q\Big\|_{L^\infty} \\
&\le \|\{s_Q\}_Q\|_{\dot{b}_p^{\alpha q}}\,\|\{t_Q\}_Q\|_{\dot{b}_\infty^{\gamma q'}},
\end{aligned}$$

by Hölder's inequality for $q \geq 1$ and the embedding $\dot{b}_p^{\alpha q} \hookrightarrow \dot{b}_p^{\alpha 1}$ for $q < 1$.

For the other direction, suppose $\ell \in [\dot{b}_p^{\alpha q}(\mathbb{C})]^*$. Because the sequences with only finitely many non-zero components are dense in $\dot{b}_p^{\alpha q}$, there exists a sequence $t = \{t_Q\}_Q$ such that $\ell(s) = \sum_Q s_Q t_Q$ for all $s \in \dot{b}_p^{\alpha q}$. Suppose first that $1 < q < \infty$. From the definitions, we get that

$$\|t\|_{\dot{b}_\infty^{\gamma q'}(\mathbb{C})} = \Big(\sum_{\nu \in \mathbb{Z}} 2^{-\nu(\alpha+n/2-n/p)q'} \sup_{Q \in \mathcal{D}_\nu} |t_Q|^{q'} \Big)^{1/q'},$$

which we want to show is bounded by $c\|\ell\|$. For each $\nu \in \mathbb{Z}$ we select one $k(\nu)$ such that $|t_{Q_{\nu k(\nu)}}| \geq \frac{1}{2} \sup_{Q \in \mathcal{D}_\nu} |t_Q|$. We define a sequence $s = \{s_Q\}_Q$ by setting $s_{Q_{\nu k(\nu)}} = 2^{-\nu(\alpha+n/2-n/p)q'} |t_{Q_{\nu k(\nu)}}|^{q'-1} \operatorname{sgn}(t_{Q_{\nu k(\nu)}})$ if $Q \in \{Q_{\nu k(\nu)}\}_{\nu \in \mathbb{Z}}$, and $s_Q = 0$ otherwise. Then

$$\begin{aligned} \|s\|_{\dot{b}_p^{\alpha q}} &= \Big(\sum_{\nu \in \mathbb{Z}} (2^{\nu(\alpha+n/2-n/p)} |s_{Q_{\nu k(\nu)}}|)^q \Big)^{1/q} \\ &= \Big(\sum_{\nu \in \mathbb{Z}} (2^{-\nu(\alpha+n/2-n/p)} |t_{Q_{\nu k(\nu)}}|)^{(q'-1)q} \Big)^{1/q} \approx ||t||_{\dot{b}_\infty^{\gamma q'}}^{q'/q}, \end{aligned}$$

by the choice of $k(\nu)$ and the relation $(q'-1)q = q'$. Hence,

$$\begin{aligned} ||t||_{\dot{b}_\infty^{\gamma q'}}^{q'} &\approx \sum_{\nu \in \mathbb{Z}} 2^{-\nu(\alpha+n/2-n/p)q'} |t_{Q_{\nu k(\nu)}}|^{q'} \\ &= \langle s, t \rangle \leq \|\ell\| \, \|s\|_{\dot{b}_p^{\alpha q}} \leq c\|\ell\| \, ||t||_{\dot{b}_\infty^{\gamma q'}}^{q'/q}. \end{aligned}$$

Dividing and using the relation $q'(1 - 1/q) = 1$ yields $\|t\|_{\dot{b}_\infty^{\gamma q'}} \leq c\|\ell\|$, as desired. (Strictly speaking, we should truncate s so that the sum over ν is finite, apply the above argument to obtain an estimate as above for the correspondingly truncated version of t, and then take the supremum over all truncations to obtain the complete estimate.)

The case $0 < q \leq 1$ is simpler. Then $\|t\|_{\dot{b}_\infty^{\gamma\infty}(\mathbb{C})} = \sup_Q(|Q|^{-\gamma/n-1/2}|t_Q|)$. Select P dyadic such that $|P|^{-\gamma/n-1/2}|t_P| \geq \frac{1}{2}\|t\|_{\dot{b}_\infty^{\gamma\infty}}$. Let $s = \{s_Q\}_Q$ satisfy $s_P = |P|^{-\gamma/n-1/2} \operatorname{sgn}(t_P)$ and $s_Q = 0$ for $Q \neq P$. Then, by the definition of γ, $\|s\|_{\dot{b}_p^{\alpha q}} = |P|^{-\alpha/n-1/2+1/p}|s_P| = 1$. Thus,

$$\|t\|_{\dot{b}_\infty^{\gamma\infty}} \approx |P|^{-\gamma/n-1/2} |t_P| = \langle s, t \rangle \leq \|\ell\| \, \|s\|_{\dot{b}_p^{\alpha q}} = \|\ell\|. \qquad \square$$

Reducing operators allow us to reduce the calculation of $[\dot{b}_p^{\alpha q}(W)]^*$ to the unweighted case just considered.

Theorem 5.3. *Let* $\alpha \in \mathbb{R}$, $0 < q < \infty$, $0 < p \le 1$, *and let* $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$ *be a sequence of reducing operators of order* p *for a matrix weight* W. *Then*

$$[\dot{b}_p^{\alpha q}(W)]^* \approx \dot{b}_\infty^{-\alpha+n(1/p-1),q'}(\{A_Q^{-1}\}),$$

with the pairing $\langle \vec{s}, \vec{t}\,\rangle = \sum_{Q\in\mathcal{D}} \langle \vec{s}_Q, \vec{t}_Q\rangle$.

Proof. By Lemma 5.1, it suffices to show

$$[\dot{b}_p^{\alpha q}(\{A_Q\})]^* \approx \dot{b}_\infty^{-\alpha+n(1/p-1),q'}(\{A_Q^{-1}\}).$$

This result follows easily from (5.2), Lemma 5.2, and the identity $\langle \vec{s}_Q, \vec{t}_Q\rangle = \langle A_Q\vec{s}_Q, A_Q^{-1}\vec{t}_Q\rangle$. ☐

Our next question is whether it is possible to express $\dot{b}_\infty^{-\alpha+n(1/p-1),q'}(\{A_Q^{-1}\})$ in terms of W. However, we need to introduce a new notation for matrix-weighted Besov spaces, since it seems we don't yet have enough indices.

Definition (uncorrelated matrix-weighted discrete Besov space)**.** Let W be a matrix weight, $\alpha \in \mathbb{R}$, $0 < q \le \infty$, $0 < r < \infty$, and $0 < p \le \infty$. The *uncorrelated* Besov space $\dot{b}_{pr}^{\alpha q}(W)$ consists of all vector-valued sequences $\vec{s}$ such that

$$\|\vec{s}\,\|_{\dot{b}_{pr}^{\alpha q}(W)} = \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{Q\in\mathcal{D}_\nu} |Q|^{-1/2} \, \|W^{1/r}(t)\vec{s}_Q\| \chi_Q(t) \right\|_{L^p(dt)} \right\}_\nu \right\|_{l^q} < \infty.$$

Note that for $p < \infty$, $\dot{b}_{pr}^{\alpha q}(W) = \dot{b}_p^{\alpha q}(W^{p/r})$, so in particular, $\dot{b}_{pp}^{\alpha q}(W) = \dot{b}_p^{\alpha q}(W)$ and

$$\dot{b}_{p'}^{-\alpha q'}(W^{-p'/p}) = \dot{b}_{p'p}^{-\alpha q'}(W^{-1}), \tag{5.4}$$

where p' is the conjugate index for p.

This notation will allow us to characterize duals of $\dot{b}_p^{\alpha q}(W)$ in terms of matrix weights. We begin with the following lemma.

Lemma 5.4. *Suppose* $0 < p \le 1$, $W \in A_p$, *and* $\{A_Q\}_Q$ *is a sequence of reducing operators of order* p *for* W. *Then for every* $Q \in \mathcal{D}$ *and every* $z \in \mathbb{C}^m$,

$$\|A_Q^{-1}z\| \approx \operatorname*{ess\,sup}_{t\in Q} \|W^{-1/p}(t)z\|,$$

with equivalence constants independent of Q *and* z.

Proof. For $z \in \mathbb{C}^m$,

$$
\begin{aligned}
\|A_Q^{-1} z\| &= \sup_{y \in \mathbb{C}^m} \frac{|\langle z, y\rangle|}{\|A_Q y\|} \\
&= \sup_{y \in \mathbb{C}^m} \frac{1}{\|A_Q y\|} \left(\int_Q |\langle W^{-1/p}(t) z, W^{1/p}(t) y\rangle|^p \frac{dt}{|Q|} \right)^{1/p} \\
&\le \operatorname*{ess\,sup}_{t \in Q} \|W^{-1/p}(t) z\| \sup_{y \in \mathbb{C}^m} \frac{\left(\int_Q \|W^{1/p}(t) y\|^p \frac{dt}{|Q|} \right)^{1/p}}{\|A_Q y\|} \\
&\approx \operatorname*{ess\,sup}_{t \in Q} \|W^{-1/p}(t) z\|.
\end{aligned}
$$

On the other hand, for a.e. $t \in Q$,

$$\|A_Q W^{-1/p}(t)\| = \|W^{-1/p}(t) A_Q\| \le ||W||_{A_p}^p,$$

since A_Q and $W^{-1/p}(t)$ are self-adjoint operators. Letting $z = A_Q y$, we obtain

$$\|W^{-1/p}(t) z\| \le ||W||_{A_p}^p \|A_Q^{-1} z\| \quad \text{for a.e. } t \in Q,$$

or, equivalently,

$$\operatorname*{ess\,sup}_{t \in Q} \|W^{-1/p}(t) z\| \le ||W||_{A_p}^p \|A_Q^{-1} z\|. \qquad \square$$

Lemma 5.5. *Let $\alpha \in \mathbb{R}$, $0 < q \le \infty$ and $0 < p \le 1$. If $W \in A_p$ and $\{A_Q\}_Q$ is a sequence of reducing operators of order p for W, then $\dot{b}^{\alpha q}_{\infty p}(W^{-1}) \approx \dot{b}^{\alpha q}_{\infty}(\{A_Q^{-1}\})$, with equivalent norms.*

Proof. By Lemma 5.4,

$$
\begin{aligned}
\Big\| \sum_{Q \in \mathcal{D}_\nu} |Q|^{-1/2} \|A_Q^{-1} \vec{s}_Q\| \chi_Q \Big\|_{L^\infty} &= \sup_{Q \in \mathcal{D}_\nu} |Q|^{-1/2} \|A_Q^{-1} \vec{s}_Q\| \\
&\approx \sup_{Q \in \mathcal{D}_\nu} |Q|^{-1/2} \operatorname*{ess\,sup}_{t \in Q} \|W^{-1/p}(t) \vec{s}_Q\| \\
&= \Big\| \sum_{Q \in \mathcal{D}_\nu} |Q|^{-1/2} \|W^{-1/p} \vec{s}_Q\| \chi_Q \Big\|_{L^\infty}.
\end{aligned}
$$

Multiplying by $2^{\nu\alpha}$ and taking the ℓ^q norm in ν yields

$$\|\vec{s}\|_{\dot{b}^{\alpha q}_{\infty}(\{A_Q^{-1}\})} \approx \|\vec{s}\|_{\dot{b}^{\alpha q}_{\infty p}(W^{-1})}. \qquad \square$$

We state the following characterization for the entire range of p.

Theorem 5.6. *Let $W \in A_p$, $0 < p < \infty$, $\alpha \in \mathbb{R}$, and $0 < q < \infty$. Then*

$$[\dot{b}_p^{\alpha q}(W)]^* \approx \dot{b}_{p'p}^{\gamma q'}(W^{-1}), \tag{5.5}$$

where $\gamma = -\alpha$ if $p \geq 1$, $\gamma = -\alpha + n(1/p - 1)$ if $p < 1$, and $p' = \infty$ for $p \leq 1$, with the pairing $\langle \vec{s}, \vec{t} \rangle = \sum_{Q \in \mathcal{D}} \langle \vec{s}_Q, \vec{t}_Q \rangle$.

Proof. For $p > 1$, (5.5) is a rephrase of Theorem A2 in [20] using a new notation or (5.4). For $0 < p \leq 1$, it follows immediately from Theorem 5.3 and Lemma 5.5. ◻

Duality of continuous Besov spaces. First we define the averaging Besov spaces for $p < 1$.

Definition (Averaging matrix-weighted Besov space $\dot{B}_p^{\alpha q}(\{A_Q\})$)**.** For $\alpha \in \mathbb{R}$, $0 < p \leq 1$, $0 < q \leq \infty$, $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$, and $\varphi \in \mathcal{A}$, the Besov space $\dot{B}_p^{\alpha q}(\{A_Q\})$ is the collection of all vector-valued distributions $\vec{f} = (f_1, \dots, f_m)^T$ with $f_i \in S'/\mathcal{P}(\mathbb{R}^n)$, $1 \leq i \leq m$, such that

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} = \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{Q \in \mathcal{D}_\nu} \|A_Q \cdot (\varphi_\nu * \vec{f})\| \chi_Q \right\|_{L^p} \right\}_\nu \right\|_{l^q} < \infty.$$

We will see that this space is well-defined (i.e., independent of $\varphi \in \mathcal{A}$), if $\{A_Q\}_Q$ is a sequence of reducing operators of order p for an a.e. invertible doubling matrix weight.

We remark that if W is an a.e. invertible doubling matrix weight of order p and $\{A_Q\}_Q$ is a sequence of reducing operators of order p for W, then $\|A_{Q_{\nu k}} A_{Q_{\nu \ell}}^{-1}\|^p \leq c(1 + |k - \ell|)^\beta$, for all $\nu \in \mathbb{Z}$ and all k, $\ell \in \mathbb{Z}^n$, where β is the doubling exponent of W, and c is independent of ν, k, and ℓ. In fact, more general estimates involving different levels hold, but we will need only to observe that for $|\nu - \mu| \leq 1$,

$$\|A_{Q_{\nu k}} A_{Q_{\mu \ell}}^{-1}\|^p \leq c(1 + 2^\nu |x_{Q_{\nu k}} - x_{Q_{\mu \ell}}|)^\beta. \tag{5.6}$$

To see (5.6), let $w_y(t)$ denote the scalar measure $\|W^{1/p}(t) y\|^p$ and observe that $\|A_Q y\|^p \approx w_y(Q)/|Q|$, by the definition of reducing operators. Hence, the estimate $\|A_{Q_{\nu k}} y\|^p \leq c(1 + 2^\nu |x_{Q_{\nu k}} - x_{Q_{\mu\ell}}|)^\beta \|A_{Q_{\mu\ell}} y\|^p$, for all $y \in \mathbb{C}^m$, follows from the uniform doubling property of w_y (see (2.3) with $\mu(t) = w_y(t)$). Letting $y = A_{Q_{\mu\ell}}^{-1} x$ yields (5.6).

The relation between the matrix-weighted Besov spaces $\dot{B}_p^{\alpha q}(W)$ and the averaging spaces $\dot{B}_p^{\alpha q}(\{A_Q\})$ is as suggested by Lemma 5.1.

Lemma 5.7. *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, and $0 < p < \infty$. If $W \in A_p$ and $\{A_Q\}_Q$ is a sequence of reducing operators of order p for W, then*

$$\dot{B}_p^{\alpha q}(W) \approx \dot{B}_p^{\alpha q}(\{A_Q\}),$$

with equivalent norms.

Proof. For $p > 1$, this is proved in [20, Section 5]. Assume $0 < p \leq 1$. Observe that for $\vec{g} \in E_\nu$,

$$\begin{aligned}
&\Big\| \sum_{Q \in \mathcal{D}_\nu} \|A_Q \vec{g}\| \chi_Q \Big\|_{L^p}^p \\
&\quad \approx \sum_{Q \in \mathcal{D}_\nu} \int_Q \int_Q \|W^{1/p}(s)\vec{g}(t)\|^p \frac{ds}{|Q|}\, dt \\
&\quad = \sum_{Q \in \mathcal{D}_\nu} \int_Q \int_Q \|W^{1/p}(s)W^{-1/p}(t)W^{1/p}(t)\vec{g}(t)\|^p \frac{ds}{|Q|}\, dt \\
&\quad \leq \|W\|_{A_p} \sum_{Q \in \mathcal{D}_\nu} \int_Q \|W^{1/p}(t)\vec{g}(t)\|^p \, dt = \|W\|_{A_p} \|\vec{g}\|_{L^p(W)}^p.
\end{aligned}$$

Taking $\vec{g} = \varphi_\nu * \vec{f}$, multiplying by $2^{\nu\alpha}$, and taking the ℓ^q norm over $\nu \in \mathbb{Z}$, yields $\dot{B}_p^{\alpha q}(W) \subseteq \dot{B}_p^{\alpha q}(\{A_Q\})$.

Similarly, for the other direction we need to prove that

$$\|\vec{g}\|_{L^p(W)} \leq c \sum_{Q \in Q_\nu} \int_Q \|A_Q \vec{g}(t)\|^p \, dt,$$

for $\vec{g} \in E_\nu$, which we do by considering the $\nu = 0$ case and then dilating. Using the identity (3.4) for $\vec{g} \in E_0$ with k replaced by t, and thus, getting a similar estimate to (3.6), we obtain

$$\begin{aligned}
\sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(t)\vec{g}(t)\|^p \, dt &\leq c \sum_{\ell \in \mathbb{Z}^n} \int_{Q_{0\ell}} \int_{Q_{0\ell}} \|W^{1/p}(t)\vec{g}(y)\|^p \, dt\, dy \\
&\leq c \sum_{Q \in \mathcal{D}_\nu} \int_Q \|A_Q \vec{g}(y)\|^p \, dy,
\end{aligned}$$

as desired, with $\nu = 0$. □

We require analogues of Theorem 3.1 and Theorem 3.4 for the averaging spaces, as follows.

Theorem 5.8. *Let $\alpha \in \mathbb{R}$, $0 < p \leq 1$, $0 < q \leq \infty$, let W be a doubling matrix of order p, $\varphi \in \mathcal{A}$, and let $\{A_Q\}_Q$ be a sequence of reducing operators of order p for W.*

(i) *Suppose* $\{m_Q\}_Q$ *is a family of smooth molecules for* $\dot{B}_p^{\alpha q}(W)$. *Then*

$$\Big\| \sum_{Q\in\mathcal{D}} \vec{s}_Q m_Q \Big\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} \le c \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}, \tag{5.7}$$

and

$$\Big\| \sum_{Q\in\mathcal{D}} \vec{s}_Q \varphi_Q \Big\|_{\dot{B}_\infty^{\alpha q}(\{A_Q^{-1}\})} \le c \|\{\vec{s}_Q\}_Q\|_{\dot{b}_\infty^{\alpha q}(\{A_Q^{-1}\})}. \tag{5.8}$$

(ii) *For* $\vec{s}_Q = \langle \vec{f}, \varphi_Q\rangle$, *we have*

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} \le c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})}, \tag{5.9}$$

and

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_\infty^{\alpha q}(\{A_Q^{-1}\})} \le c \|\vec{f}\|_{\dot{B}_\infty^{\alpha q}(\{A_Q^{-1}\})}. \tag{5.10}$$

Proof. The proofs of (5.7) and (5.9) follow along the same lines as Lemmas 4.4 and 4.6 and Corollaries 4.5 and 4.8 in [20], with minor modifications. We will only require (5.7) and (5.9) for the case where $W \in A_p$, in which case they follow immediately from Theorems 3.1 and 3.4 and Lemmas 5.1 and 5.7.

To prove (5.8), define $B_\nu = \sup_{Q\in\mathcal{D}_\nu} 2^{\nu\alpha}|Q|^{-1/2}\|A_Q^{-1}\vec{s}_Q\|$, so that

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_\infty^{\alpha q}(\{A_Q^{-1}\})} = \|\{B_\nu\}_{\nu\in\mathbb{Z}}\|_{\ell^q}.$$

Then for $x \in Q$, where $\ell(Q) = 2^{-\nu}$, we have

$$2^{\nu\alpha}\Big\|A_Q^{-1}\Big(\varphi_\nu * \sum_{P\in\mathcal{D}} \vec{s}_P\varphi_P\Big)(x)\Big\| = 2^{\nu\alpha}\Big\|\sum_{\mu=\nu-1}^{\nu+1}\sum_{P\in\mathcal{D}_\mu} A_Q^{-1}\vec{s}_P(\varphi_\nu * \varphi_P)(x)\Big\|,$$

because $\varphi_\nu * \varphi_P = 0$ unless $\ell(P) = 2^{-\mu}$ satisfies $|\mu-\nu| \le 1$, by the support assumption on $\hat{\varphi}$ for $\varphi \in \mathcal{A}$. Since $\varphi \in S$, we obtain the estimate $|(\varphi_\nu * \varphi_P)(x)| \le c_M|P|^{-1/2}(1+2^\mu|x-x_P|)^{-M}$, for any $M > 0$, where $x_P = 2^{-\mu}k$ if $P = Q_{\mu k}$. Hence, for $x \in Q$ and M taken sufficiently large,

$$\begin{aligned}
&2^{\nu\alpha}\Big\|\sum_{\mu=\nu-1}^{\nu+1}\sum_{P\in\mathcal{D}_\mu} A_Q^{-1}\vec{s}_P(\varphi_\nu * \varphi_P)(x)\Big\| \\
&\quad\le c\sum_{\mu=\nu-1}^{\nu+1} 2^{(\nu-\mu)\alpha}\sum_{P\in\mathcal{D}_\mu}\|A_Q^{-1}A_P\|2^{\mu\alpha}|P|^{-1/2}\|A_P^{-1}\vec{s}_P\|(1+2^\mu|x_Q-x_P|)^{-M} \\
&\quad\le c\sum_{\mu=\nu-1}^{\nu+1} B_\mu \sum_{P\in\mathcal{D}_\mu}\|A_Q^{-1}A_P\|(1+2^\mu|x_Q-x_P|)^{-M} \le c(B_{\nu-1}+B_\nu+B_{\nu+1}),
\end{aligned}$$

by (5.6). Therefore,

$$2^{\nu\alpha}\sup_{Q\in\mathcal{D}_\nu}\operatorname*{ess\,sup}_{x\in Q}\left\|A_Q^{-1}\Big(\varphi_\nu * \sum_{P\in\mathcal{D}}\vec{s}_P\varphi_P\Big)(x)\right\| \le c(B_{\nu-1}+B_\nu+B_{\nu+1}).$$

Taking the ℓ^q norm over $\nu\in\mathbb{Z}$ yields (5.8).

For (5.10), note that $\langle\vec{f},\varphi_Q\rangle = |Q|^{1/2}(\tilde{\varphi}_\nu * \vec{f}\,)(x_Q)$, where $\tilde{\varphi}_\nu(x)\equiv\overline{\varphi_\nu(-x)}$. Hence, the continuity of $\tilde{\varphi}_\nu * \vec{f}$ guarantees that

$$\sup_{Q\in\mathcal{D}_\nu}\|A_Q^{-1}(\tilde{\varphi}_\nu * \vec{f}\,)(x_Q)\| \le \sup_{Q\in\mathcal{D}_\nu}\operatorname*{ess\,sup}_{x\in Q}\|A_Q^{-1}(\tilde{\varphi}_\nu * \vec{f}\,)(x)\|.$$

Multiplying by $2^{\nu(\alpha+n/2)}$ and taking the ℓ^q norm over $\nu\in\mathbb{Z}$ yields

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_\infty^{\alpha q}(\{A_Q^{-1}\})} \le \|\vec{f}\,\|_{\dot{B}_\infty^{\alpha q}(\{A_Q^{-1}\},\tilde{\varphi})},$$

where the last norm is in the space defined with respect to the admissible kernel $\tilde{\varphi}$. Using (5.8) as in [19, pp. 292-293], we can replace $\tilde{\varphi}$ with φ and obtain (5.10). □

A more general version of (5.8), with appropriate molecules m_Q in place of φ_Q, can be proved with some extra effort, but we will not need this result. Also, as in [19, pp. 292-293], we obtain the independence of the spaces $\dot{B}_p^{\alpha q}(\{A_Q\})$ from the choice of $\varphi\in\mathcal{A}$.

Theorem 5.9. *Let $0<p\le 1$, $0<q<\infty$, $\alpha\in\mathbb{R}$, $W\in A_p$, and let $\{A_Q\}_Q$ be a sequence of reducing operators of order p for W. Then*

$$[\dot{B}_p^{\alpha q}(W)]^* \approx \dot{B}_\infty^{-\alpha+n(1/p-1),q'}(\{A_Q^{-1}\}),$$

with the pairing $\langle\vec{f},\vec{g}\,\rangle = \int_{\mathbb{R}^n}\langle\vec{f}(x),\vec{g}(x)\rangle\,dx$.

Proof. By Lemma 5.7, $[\dot{B}_p^{\alpha q}(W)]^* \approx [\dot{B}_p^{\alpha q}(\{A_Q\})]^*$. Let $\gamma = -\alpha+n(1/p-1)$. We first prove the embedding $[\dot{B}_p^{\alpha q}(\{A_Q\})]^* \subseteq \dot{B}_\infty^{\gamma q'}(\{A_Q^{-1}\})$.

Let $\ell\in[\dot{B}_p^{\alpha q}(\{A_Q\})]^*$. For each Q dyadic and $i=1,2,\ldots,m$, let $\vec{\psi}_Q^{(i)}$ be the vector-valued function whose i^{th} component is ψ_Q and all of whose other components are 0. Define $\vec{\varphi}_Q^{(i)}$ in the same way, with φ in place of ψ. Let $t_Q^{(i)} = \overline{\ell(\vec{\psi}_Q^{(i)})}$ and let $\vec{t}_Q = (t_Q^{(1)}, t_Q^{(2)},\ldots,t_Q^{(m)})^T$. Then for $\vec{f}\in\dot{B}_p^{\alpha q}(\{A_Q\})$, the φ-transform identity reads

$$\vec{f} = \sum_{Q,i}\langle\vec{f},\vec{\varphi}_Q^{(i)}\rangle\vec{\psi}_Q^{(i)}.$$

Hence, taking ℓ inside the sum, which is permissible because the sum converges in norm,

$$\begin{aligned}\ell(\vec{f}) &= \sum_{Q,i}\langle \vec{f}, \vec{\varphi}_Q^{(i)}\rangle \ell(\vec{\psi}_Q^{(i)}) = \sum_{Q,i}\langle \vec{f}, \vec{\varphi}_Q^{(i)}\rangle \overline{t_Q^{(i)}} \\ &= \Big\langle \vec{f}, \sum_{Q,i} t_Q^{(i)} \vec{\varphi}_Q^{(i)}\Big\rangle = \Big\langle \vec{f}, \sum_{Q\in\mathcal{D}} \vec{t}_Q \varphi_Q\Big\rangle.\end{aligned}$$

Therefore, ℓ is represented by the distribution $\vec{g} = \sum_{Q\in\mathcal{D}} \vec{t}_Q \varphi_Q$.

Letting $\vec{t} = \{\vec{t}_Q\}_Q$, we have

$$\|g\|_{\dot{B}_\infty^{\gamma q'}(\{A_Q^{-1}\})} \le C\|\vec{t}\|_{\dot{b}_\infty^{\gamma q'}(\{A_Q^{-1}\})},$$

by (5.8). By the duality $[\dot{b}_p^{\alpha q}(\{A_Q\})]^* \approx \dot{b}_\infty^{\gamma q'}(\{A_Q^{-1}\})$ (see Lemma 5.1 and Theorem 5.3),

$$\|\vec{t}\|_{\dot{b}_\infty^{\gamma q'}(\{A_Q^{-1}\})} \le C \cdot \sup_{\|\vec{s}\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}\le 1} |\langle \vec{s}, \vec{t}\rangle|.$$

But

$$\begin{aligned}|\langle \vec{s}, \vec{t}\rangle| &= \sum_{Q\in\mathcal{D}} \langle \vec{s}_Q, \vec{t}_Q\rangle = \Big|\sum_{Q,i} s_Q^{(i)} \overline{t_Q^{(i)}}\Big| = \Big|\sum_{Q,i} s_Q^{(i)} \ell(\vec{\psi}_Q^{(i)})\Big| \\ &= \Big|\ell\Big(\sum_{Q,i} s_{Q,i} \vec{\psi}_Q^{(i)}\Big)\Big| = \Big|\ell\Big(\sum_{Q\in\mathcal{D}} \vec{s}_Q \psi_Q\Big)\Big| \\ &\le \|\ell\| \Big\|\sum_{Q\in\mathcal{D}} \vec{s}_Q \psi_Q\Big\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} \le C\|\ell\|\,\|\vec{s}\|_{\dot{b}_p^{\alpha q}(\{A_Q\})},\end{aligned}$$

using (5.7). Thus, we have that $g \in \dot{B}_\infty^{\gamma q'}(\{A_Q^{-1}\})$ with $\|g\|_{\dot{B}_\infty^{\gamma q'}(\{A_Q^{-1}\})} \le c\|\ell\|$, as desired.

We now prove $\dot{B}_\infty^{\gamma q'}(\{A_Q^{-1}\}) \subseteq [\dot{B}_p^{\alpha q}(\{A_Q\})]^*$. Suppose $\vec{f} \in \dot{B}_p^{\alpha q}(\{A_Q\})$ and $\vec{g} \in \dot{B}_\infty^{\gamma q'}(\{A_Q^{-1}\})$. Then, using Lemma 5.1 and Theorem 5.3, we get

$$\begin{aligned}|\langle \vec{f}, \vec{g}\rangle| &= \Big|\Big\langle \sum_{Q\in\mathcal{D}} \langle \vec{f}, \varphi_Q\rangle \psi_Q, \vec{g}\Big\rangle\Big| = \Big|\sum_{Q\in\mathcal{D}} \langle\langle \vec{f}, \varphi_Q\rangle, \langle \vec{g}, \psi_Q\rangle\rangle\Big| \\ &\le C\|\{\langle \vec{f}, \varphi_Q\rangle\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}\,\|\{\langle \vec{g}, \psi_Q\rangle\}_Q\|_{\dot{b}_\infty^{\gamma q'}(\{A_Q^{-1}\})} \\ &\le C\|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})}\,\|\vec{g}\|_{\dot{B}_\infty^{\gamma q'}(\{A_Q^{-1}\})},\end{aligned}$$

by (5.9) and (5.10) (applied with ψ in place of φ), as desired. □

As in the discrete case, we need to extend the Besov space notation to four indices to express the dual of $\dot{B}_p^{\alpha q}(W)$ explicitly in terms of W.

Definition (uncorrelated matrix-weighted continuous Besov space)**.** Let $\alpha \in \mathbb{R}$, $0 < q \le \infty$, $0 < r < \infty$, $0 < p \le \infty$, and let W be a matrix weight. Then the *uncorrelated* Besov space $\dot{B}_{pr}^{\alpha q}(W)$ is the collection of all vector-valued distributions $\vec{f}$ with $f_i \in S'/\mathcal{P}(\mathbb{R}^n)$, $1 \le i \le m$, such that

$$\|\vec{f}\|_{\dot{B}_{p\,r}^{\alpha\,q}(W)} = \left\|\left\{2^{\nu\alpha}\left\|W^{1/r}\cdot\left(\varphi_\nu * \vec{f}\right)\right\|_{L^p}\right\}_\nu\right\|_{l^q} < \infty.$$

What is important for us is the following relation.

Lemma 5.10. *Let $\alpha \in \mathbb{R}$, $0 < q \le \infty$ and $0 < p \le 1$. If $W \in A_p$ and $\{A_Q\}_Q$ is a sequence of reducing operators of order p for W, then*

$$\dot{B}_{\infty p}^{\alpha q}(W^{-1}) \approx \dot{B}_{\infty}^{\alpha q}(\{A_Q^{-1}\}).$$

Proof. As noted in the proof of Lemma 5.4, for almost every $t \in Q$, we have

$$\|W^{-1/p}(t)A_Q\| = \|A_Q W^{-1/p}(t)\| \le \|W\|_{A_p}.$$

Hence, for a.e. $t \in Q$,

$$\begin{aligned}\|W^{-1/p}(t)(\varphi_\nu * \vec{f})(t)\| &= \|W^{-1/p}(t)A_Q A_Q^{-1}(\varphi_\nu * \vec{f})(t)\| \\ &\le \|W\|_{A_p}\|A_Q^{-1}(\varphi_\nu * \vec{f})(t)\|.\end{aligned}$$

Hence,

$$\begin{aligned}\|W^{-1/p}(\varphi_\nu * \vec{f})\|_{L^\infty} &= \sup_{Q\in\mathcal{D}_\nu}\ \operatorname*{ess\,sup}_{t\in Q}\|W^{-1/p}(t)(\varphi_\nu * \vec{f})(t)\| \\ &\le \|W\|_{A_p}\sup_{Q\in\mathcal{D}_\nu}\ \operatorname*{ess\,sup}_{t\in Q}\|A_Q^{-1}(\varphi_\nu * \vec{f})(t)\| \\ &= \|W\|_{A_p}\left\|\sum_{Q\in\mathcal{D}_\nu}\|A_Q^{-1}(\varphi_\nu * \vec{f})\|\chi_Q\right\|_{L^\infty}.\end{aligned}$$

Multiplying by $2^{\nu\alpha}$ and taking the ℓ^q norm in ν yields

$$\|\vec{f}\|_{\dot{B}_{\infty p}^{\alpha q}(W^{-1})} \le \|W\|_{A_p}\,\|\vec{f}\|_{\dot{B}_{\infty}^{\alpha q}(\{A_Q^{-1}\})}.$$

For the opposite embedding we use Lemma 5.4 and (3.5) to get, for $\vec{g} \in E_0$,

$$\Big\| \sum_{Q\in\mathcal{D}_0} A_Q^{-1}\vec{g}\,\chi_Q \Big\|_{L^\infty}^p$$
$$= \sup_{\ell\in\mathbb{Z}^n}\sup_{s\in Q_{0\ell}} \|A_{Q_{0\ell}}^{-1}\vec{g}(s)\|^p \le \sup_{\ell\in\mathbb{Z}^n}\sup_{s\in Q_{0\ell}} \operatorname*{ess\,sup}_{t\in Q_{0\ell}} \|W^{-1/p}(t)\vec{g}(s)\|^p$$
$$\le c \sup_{\ell\in\mathbb{Z}^n}\sup_{s\in Q_{0\ell}} \operatorname*{ess\,sup}_{t\in Q_{0\ell}} \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m-s|)^M}\int_{Q_{0m}} \|W^{-1/p}(t)\vec{g}(y)\|^p\,dy$$
$$\le c \sup_{\ell\in\mathbb{Z}^n}\sup_{s\in Q_{0\ell}} \operatorname*{ess\,sup}_{t\in Q_{0\ell}} \sum_{m\in\mathbb{Z}^n} \frac{(1+|m-\ell|)^\beta}{(1+|m-s|)^M}\int_{Q_{0\ell}} \|W^{-1/p}(t)\vec{g}(y)\|^p\,dy$$
$$\le c \sup_{\ell\in\mathbb{Z}^n} \operatorname*{ess\,sup}_{t\in Q_{0\ell}} \sum_{m\in\mathbb{Z}^n} (1+|m-l|)^{\beta-M} \|W^{-1/p}(y)\vec{g}(y)\|_{L^\infty}^p \times \int_{Q_{0\ell}} \|W^{-1/p}(t)W^{1/p}(y)\|^p\,dy$$
$$\le c\|W^{-1/p}\vec{g}\|_{L^\infty}^p \sup_{\ell\in\mathbb{Z}^n} \operatorname*{ess\,sup}_{t\in Q_{0\ell}} \int_{Q_{0\ell}} \|W^{-1/p}(t)W^{1/p}(y)\|^p\,dy$$
$$\le c\|W\|_{A_p}\|W^{-1/p}\vec{g}\|_{L^\infty}^p,$$

by shifting Q_{0m} to $Q_{0\ell}$, using (2.3), and taking $M > \beta+n$. By dilation, we obtain a similar estimate for $\vec{g} = \varphi_\nu * \vec{f} \in E_\nu$. Multiplying by $2^{\nu\alpha}$ and taking the ℓ^q norm over $\nu \in \mathbb{Z}$ yields the result. □

Remark 5.11. With the same assumptions as in the above lemma, we also have $\dot{B}^{\alpha q}_{\infty p}(W) \approx \dot{B}^{\alpha q}_{\infty}(\{A_Q\})$. (The proof is similar to the proof of Lemmas 5.1 and 5.3 in [20].)

Obviously Theorem 5.9 and Lemma 5.10 imply the following result, which expresses the dual of $\dot{B}^{\alpha q}_p(W)$ in terms of a space defined directly in terms of W, instead of indirectly via the reducing operators associated to W.

Theorem 5.12. *Let $W \in A_p$, $0 < p \le 1$, $0 < q < \infty$, and $\alpha \in \mathbb{R}$. Then*

$$[\dot{B}^{\alpha q}_p(W)]^* \approx \dot{B}^{-\alpha+n(1/p-1),q'}_{\infty p}(W^{-1}),$$

with the pairing $\langle \vec{f}, \vec{g}\rangle = \int_{\mathbb{R}^n}\langle \vec{f}(x), \vec{g}(x)\rangle\,dx$.

6. INHOMOGENEOUS BESOV SPACES FOR $0 < p \le 1$

Results analogous to those in this paper for the homogeneous Besov spaces $\dot{B}^{\alpha q}_p(W)$ hold for the inhomogeneous Besov spaces $B^{\alpha q}_p(W)$. The results either follow from the results in the inhomogeneous case, or are proved via standard modifications of the proofs in this paper, along the lines of [19, Section 11].

A pair of kernels (φ, Φ) is admissible, $(\varphi, \Phi) \in \mathcal{A}'$, if $\varphi \in \mathcal{A}$ and $\Phi \in S(\mathbb{R}^n)$ satisfies $\operatorname{supp} \hat{\Phi} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$ and $|\hat{\Phi}(\xi)| \geq c > 0$ if $|\xi| \leq \frac{5}{3}$. For $(\varphi, \Phi) \in \mathcal{A}'$, $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p \leq 1$ (the case $p > 1$ is covered in [19, Section 11]), and $W \in A_p$, we define the inhomogeneous matrix weighted Besov space $B_p^{\alpha q}(W)$ to be the set of all $\vec{f} = (f^{(1)}, f^{(2)}, \ldots, f^{(m)})^T$ with each $f^{(i)} \in S'(\mathbb{R}^n)$, such that

$$\|\vec{f}\|_{B_p^{\alpha q}(W)} = \|\Phi * \vec{f}\|_{L^p(W)} + \Big(\sum_{\nu=1}^{\infty} (2^{\nu\alpha} \|\varphi_\nu * \vec{f}\|_{L^p(W)})^q \Big)^{1/q} < \infty.$$

We consider sequences of the form $\vec{s} = \{\vec{s}_Q\}_{Q \in \mathcal{D}, \ell(Q) \leq 1}$. For $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p \leq 1$, and $W \in A_p$, we define the discrete matrix-weighted Besov space $b_p^{\alpha q}(W)$ by restricting the ℓ^q norm in the definition of $\dot{b}_p^{\alpha q}(W)$ to $\nu \geq 0$, as in [19, Definition 11.2]. The *inhomogeneous* φ-transform is the map taking $\vec{f}$ with components in $S'(\mathbb{R}^n)$ to $\{\vec{s}_Q(\vec{f})\}_{Q \in \mathcal{D}, \ell(Q) \leq 1}$, where $\vec{s}_Q(\vec{f})$ is as above for $\ell(Q) < 1$, and $\vec{s}_Q(\vec{f}) = \langle \vec{f}, \Phi_Q \rangle$ if $\ell(Q) = 1$, where $\Phi_Q(x) = \Phi(x - k)$ for $Q = Q_{0k}$.

Theorem 6.1. *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p \leq 1$, and $W \in A_p$. Then*

$$\|\vec{f}\|_{B_p^{\alpha q}(W)} \approx \|\{\vec{s}_Q(\vec{f})\}_Q\|_{b_p^{\alpha q}(W)}. \tag{6.1}$$

The spaces $B_p^{\alpha q}(W)$, for $W \in A_p$, are well-defined (i.e., independent of the choice of $(\varphi, \Phi) \in \mathcal{A}'$).

For $p \leq 1$, we do not expect Calderón-Zygmund operators to be bounded on $B_p^{\alpha q}(W)$, due to the term $\|\Phi * \vec{f}\|_{L^p(W)}$ in the definition of the norm, and the fact that such operators are not L^p-bounded (even in the scalar, unweighted case) for $p \leq 1$. So we do not have an analogue of Theorem 4.2.

The $b_p^{\alpha q}(\{A_Q\})$ and $b_{pr}^{\alpha q}(W)$ norms of a sequence $\vec{s} = \{\vec{s}_Q\}_{Q \in \mathcal{D}, \ell(Q) \leq 1}$ are defined as for $\dot{b}_p^{\alpha q}(\{A_Q\})$ and $\dot{b}_{pr}^{\alpha q}(W)$, except that the ℓ^q-norm is taken over $\nu \geq 0$. To define $B_p^{\alpha q}(\{A_Q\})$ and $B_{pr}^{\alpha q}(W)$, we consider $\vec{f} = (f_1, \ldots, f_m)^T$ with $f_i \in S'(\mathbb{R}^n)$, we replace φ_0 with Φ, and take the ℓ^q-norm only over $\nu \geq 0$ in the definitions of $\dot{B}_p^{\alpha q}(\{A_Q\})$ and $\dot{B}_{pr}^{\alpha q}(W)$.

Theorem 6.2. *Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $0 < p \leq 1$, $W \in A_p$, and let $\{A_Q\}_{Q \in \mathcal{D}, \ell(Q) \leq 1}$ be a sequence of reducing operators of order p for W. Then*

$$[b_p^{\alpha q}(W)]^* \approx b_\infty^{-\alpha + n(1/p-1), q'}(\{A_Q^{-1}\}) \approx b_{\infty p}^{-\alpha + n(1/p-1), q'}(W^{-1}),$$

with the pairing $\langle \vec{s}, \vec{t} \rangle = \sum_{Q \in \mathcal{D}, \ell(Q) \leq 1} \langle \vec{s}_Q, \vec{t}_Q \rangle$, and

$$[B_p^{\alpha q}(W)]^* \approx B_\infty^{-\alpha + n(1/p-1), q'}(\{A_Q^{-1}\}) \approx B_{\infty p}^{-\alpha + n(1/p-1), q'}(W^{-1}),$$

with the pairing $\langle \vec{f}, \vec{g} \rangle = \int_{\mathbb{R}^n} \langle \vec{f}(x), \vec{g}(x) \rangle \, dx$.

7. INSUFFICIENCY OF THE DOUBLING CONDITION

In this section we give an example which shows that the doubling condition is not in general sufficient to guarantee the norm equivalence (1.5). This example was shown to us by Fedor Nazarov.

Theorem 7.1. *Define the matrix weight* W *on* $\mathbb{R}$ *with values in the* 2×2 *non-negative definite matrices by*

$$W(t) = \begin{pmatrix} \cos^2 t & \cos t \sin t \\ \cos t \sin t & \sin^2 t \end{pmatrix}.$$

Then W *satisfies the doubling condition of order* p *for all* $p \in (0, \infty)$, *but there exist* $\varphi \in \mathcal{A}$ *and* $f \in S(\mathbb{R})$ *such that the inequality*

$$\|\{\langle f, \varphi_Q \rangle\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \le c \|f\|_{\dot{B}_p^{\alpha q}(W)} \tag{7.1}$$

fails for every $\alpha \in \mathbb{R}$, $0 < q \le \infty$, *and* $0 < p < \infty$.

Proof. A calculation shows that for any $z = \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] \in \mathbb{C}^2$, we have

$$\langle W(t)z, z \rangle = |\langle z, e_t \rangle|^2, \quad \text{where } e = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \tag{7.2}$$

which demonstrates the non-negative definiteness of $W(t)$. Note that $W(t)$ is unitarily similar to the matrix $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$; in fact $W(t) = U_t A U_t^{-1}$ where

$$U_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

It follows that $W(t)^r = W(t)$ for every $r > 0$.

To see that W is doubling of order p, we can assume y is real (because W is real) and $\|y\| = 1$ in (2.1), say $y = (\sin \alpha, \cos \alpha)^T$. Then

$$\begin{aligned} \|W^{1/p}(t)y\|^2 = \|W^{1/2}(t)y\|^2 &= \langle W(t)y, y \rangle = |\langle y, e_t \rangle|^2 \\ &= |y_1 \cos t + y_2 \sin t|^2 = |\sin(t + \alpha)|^2. \end{aligned}$$

Thus, (2.1) reduces to $\int_{B_{2\delta}(z)} |\sin(t + \alpha)|^p \, dt \le c \int_{B_\delta(z)} |\sin(t + \alpha)|^p \, dt$. Since z is arbitrary, we may assume that $\alpha = 0$, and this amounts to showing that $w(t) = |\sin t|^p$ is a doubling measure on $\mathbb{R}$. This is elementary: we only need to consider δ very small, and then we need only consider z very close to 0. But in this region $\sin t$ is well approximated by t, and the estimates follow from the standard fact that $w(t) = |t|^p$ is doubling (which holds for $p > -1$). From the case $z = 0$ and $\delta \to 0$, we see that the doubling constant of order p is at least 2^{p+1}.

We will select our test functions φ and f such that $\varphi_\nu * f = 0$ for $\nu \neq 0$, hence $\|f\|_{\dot{B}_p^{\alpha q}(W)}$ reduces to $\|\varphi_0 * f\|_{L^p(W)} = \|W^{1/p}(\varphi_0 * f)\|_{L^p(\mathbb{R})}$. We will choose f so that

$$\varphi_0 * f = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \gamma(t),$$

where $\gamma \in S(\mathbb{R})$. Then since $W^{1/p} = W$, a direct calculation shows that $W^{1/p}(t)(\varphi_0 * f)(t) = 0$ for every t (or, note that

$$\begin{aligned} \|W^{1/p}(\varphi_0 * f)\|^2 &= \|W^{1/2}(\varphi_0 * f)\|^2 = \langle W^{1/2}(\varphi_0 * f), W^{1/2}(\varphi_0 * f)\rangle \\ &= \langle W(\varphi_0 * f), (\varphi_0 * f)\rangle = |\langle(\varphi_0 * f), e_t\rangle|^2 = 0, \end{aligned}$$

by (7.2)). Hence, the right side of (7.1) is 0.

We will select φ such that $\varphi = \tilde{\varphi}$, so the p^{th} power of the left side of (7.1) will reduce to $\sum_{k\in\mathbb{Z}} \int_k^{k+1} \|W^{1/p}(t)(\varphi_0 * f)(k)\|^p \, dt$, because $\langle f, \varphi_{Q_{\nu k}}\rangle = 2^{-\nu/2}(\tilde{\varphi}_\nu * f)(2^{-\nu}k) = 2^{-\nu/2}(\varphi_\nu * f)(2^{-\nu}k)$, which is 0 for $\nu \neq 0$. Notice that

$$\begin{aligned} \|W^{1/p}(t)(\varphi_0 * f)(k)\|^2 &= \|W^{1/2}(t)(\varphi_0 * f)(k)\|^2 \\ &= \langle W(t)(\varphi_0 * f)(k), (\varphi_0 * f)(k)\rangle \\ &= \left| \left\langle \begin{bmatrix} -\sin k \\ \cos k \end{bmatrix} \gamma(k), e_t \right\rangle \right|^2 \\ &= |(-\sin k \cos t + \cos k \sin t)\gamma(k)|^2 \\ &= |\sin(t-k)\gamma(k)|^2. \end{aligned}$$

Hence, the p^{th} power of the left side of (7.1) is

$$\sum_{k\in\mathbb{Z}} \int_k^{k+1} |\sin(t-k)\gamma(k)|^p \, dt = \sum_{k\in\mathbb{Z}} |\gamma(k)|^p \int_0^1 |\sin t|^p \, dt.$$

Since γ is of small exponential type and not identically 0, we must have $\gamma(k) \neq 0$ for some k (γ is determined by $\{\gamma(k)\}_{k\in\mathbb{Z}}$ by the Shannon sampling formula, as in the first line of the proof of Lemma 3.3). Hence, the left side of (7.1) is non-zero.

It remains to choose φ and f as described. We can select φ so that $\hat{\varphi}$ is real (hence $\tilde{\varphi} = \varphi$) and $\hat{\varphi}$ is identically 1 in $I_{-1} = (-1-\delta, -1+\delta)$ and $I_1 = (1-\delta, 1+\delta)$ for sufficiently small δ. From the condition $\sum_\nu |\hat{\varphi}(2^\nu\xi)|^2 = 1$, it follows that $\widehat{\varphi_\nu} = 0$ on I_{-1} and I_1 for $\nu \neq 0$. We let $\gamma \in S$ be such that $\hat{\gamma}$ is supported in $(-\delta, \delta)$. We then set

$$f = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \gamma(t).$$

Then the second component f_2 of f satisfies

$$\hat{f}_2(\xi) = (\cos t\gamma(t))\hat{}(\xi) = \frac{1}{2}[(e^{it} + e^{-it})\gamma(t)]\hat{}(\xi) = \frac{1}{2}(\hat{\gamma}(\xi - 1) + \hat{\gamma}(\xi + 1)).$$

Hence, $\hat{f}_2$ is supported in $I_{-1} \cup I_1$, so $\varphi_0 * f_2 = f_2 = \gamma(t)\cos t$, as desired, and $\varphi_\nu * f_2 = 0$ for $\nu \neq 0$. The argument for the first component of f is similar. □

Observe that the matrix $W(t)$ is singular for every t, hence, $W \notin A_p$ for every p. Moreover, because the doubling exponent β_p of order p for W is at least $p + 1$, we see that the condition $p > \beta_p$ of Theorem 1.6 in [19] fails in this example for all p.

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REFERENCES

[1] O.V. BESOV, *On embedding and extension theorems for some function classes*, Trudy Mat. Inst. Steklov **60** (1960), 42–81. (Russian)

[2] C. CARATHÉODORY, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annahmen*, Math. Ann. **64** (1907), 95–115.

[3] INGRID DAUBECHIES, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **41** (1988), 909–996. MR 90m:42039

[4] MICHAEL FRAZIER and BJÖRN JAWERTH, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34** (1985), 777–799. MR 87h:46083

[5] ______, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. **93** (1990), 34–170. MR 92a:46042

[6] MICHAEL FRAZIER, BJÖRN JAWERTH, and GUIDO WEISS, *Littlewood-Paley theory and the study of function spaces*, CBMS Regional Conference Series in Mathematics, vol. 79, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1991, ISBN 0-8218-0731-5. MR 92m:42021

[7] M. FRAZIER, R. TORRES, and G. WEISS, *The boundedness of Calderón-Zygmund operators on the spaces* $\dot{F}_p^{\alpha,q}$, Rev. Mat. Iberoamericana **4** (1988), 41–72. MR 90k:42029

[8] JOSÉ GARCÍA-CUERVA and JOSÉ L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985, ISBN 0-444-87804-1. MR 87d:42023

[9] Richard Hunt, Benjamin Muckenhoupt, and Richard Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227–251. MR 47 #701

[10] Fritz John, *Extremum problems with inequalities as subsidiary conditions*, Studies and Essays Presented to R. Courant on His 60th Birthday, January 8, 1948, Interscience Publishers, Inc., New York, N. Y., 1948, pp. 187–204. MR 10,719b

[11] P. G. Lemarié and Y. Meyer, *Ondelettes et bases hilbertiennes*, Rev. Mat. Iberoamericana **2** (1986), 1–18. 864 650 (French)

[12] Yves Meyer, *Principe d'incertitude, bases hilbertiennes et algèbres d'opérateurs*, Astérisque **145-146** (1987), 4, 209–223. MR 88g:42012 (French)

[13] F. L. Nazarov and S. R. Treil, *The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis*, Algebra i Analiz **8** (1996), 32–162. MR 99d:42026 (Russian, with Russian summary)

[14] F. Nazarov, S. Treil, and A. Volberg, *The Bellman functions and two-weight inequalities for Haar multipliers*, J. Amer. Math. Soc. **12** (1999), 909–928, http://dx.doi.org/10.1090/S0894-0347-99-00310-0. MR 2000k:42009

[15] Jaak Peetre, *New thoughts on Besov spaces*, Mathematics Department, Duke University, Durham, N.C., 1976. MR 57 #1108

[16] Vladimir V. Peller, *Hankel operators and their applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003, ISBN 0-387-95548-8. MR 2004e:47040

[17] Gilles Pisier, *The volume of convex bodies and Banach space geometry*, Cambridge Tracts in Mathematics, vol. 94, Cambridge University Press, Cambridge, 1989, ISBN 0-521-36465-5, 0-521-66635-X. MR 91d:52005

[18] Svetlana Roudenko, *The theory of function spaces with matrix weights*, Ph. D. thesis, Mich. State Univ., 2002.

[19] ______, *Matrix-weighted Besov spaces*, Trans. Amer. Math. Soc. **355** (2003), 273–314 (electronic), http://dx.doi.org/10.1090/S0002-9947-02-03096-9. MR 2003k:42037

[20] ______, *Duality of matrix-weighted Besov spaces*, Studia Math. **160** (2004), 129–156, http://dx.doi.org/10.1090/S0002-9947-02-03096-9. 2 033 147

[21] Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. MR 44 #7280

[22] ______, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, ISBN 0-691-03216-5. MR 95c:42002

[23] Mitchell H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space. I. Principal properties*, J. Math. Mech. **13** (1964), 407–479. MR 29 #462

[24] ______, *On the theory of Lipschitz spaces of distributions on Euclidean n-space. II. Translation invariant operators, duality, and interpolation*, J. Math. Mech. **14** (1965), 821–839. MR 31 #5087

[25] ______, *On the theory of Lipschitz spaces of distributions on Euclidean n-space. III. Smoothness and integrability of Fourier tansforms, smoothness of convolution kernels*, J. Math. Mech. **15** (1966), 973–981. MR 33 #6381

[26] Hans Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983, ISBN 3-7643-1381-1. MR 86j:46026

[27] Sergei Treil and Alexander Volberg, *Wavelets and the angle between past and future*, J. Funct. Anal. **143** (1997), 269–308, http://dx.doi.org/10.1006/jfan.1996.2986. MR 99k:42073

[28] ______, *Continuous frame decomposition and a vector Hunt-Muckenhoupt-Wheeden theorem*, Ark. Mat. **35** (1997), 363–386. MR 98m:42026

[29] A. VOLBERG, *Matrix A_p weights via S-functions*, J. Amer. Math. Soc. **10** (1997), 445–466, http://dx.doi.org/10.1090/S0894-0347-97-00233-6. MR 98a:42013

[30] ROGER WEBSTER, *Convexity*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1994, ISBN 0-19-853147-8. MR 98h:52001

[31] A. ZYGMUND, *On the preservation of classes of functions*, J. Math. Mech. **8** (1959), 889–895. MR 22 #8277

[32] ______, *On the preservation of classes of functions; eratum*, J. Math. Mech. **9** (1959), 663.

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