Global behavior of solutions to the focusing 3d cubic nonlinear Schrödinger equation[1]

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Abstract. We consider solutions u to the 3d nonlinear Schrödinger equation $i\partial_t u + \Delta u + |u|^2 u = 0$. In particular, we are interested in finding criteria on the initial data u_0 that predict the asymptotic behavior of $u(t)$: whether $u(t)$ blows-up in finite time, exists globally in time but behaves like a linear solution for large times (scatters), or exists globally in time but does not scatter. We review how this question has been resolved for H^1 data when $M[u]E[u] \le M[Q]E[Q]$, where $M[u]$ and $E[u]$ denote the mass and energy of u, and Q denotes the ground state solution to $-Q + \Delta Q + |Q|^2 Q = 0$. Then we consider the complementary case $M[u]E[u] > M[Q]E[Q]$, for which few analytical results are currently available. We start with presenting an analytical result due to Lushnikov [8] that gives a sufficient condition for blow-up, different from the previously known blow up criteria, and then present an alteration to his argument that in some cases improves upon his condition. The last condition is also extended to radial initial-data of infinite variance.

Keywords: Nonlinear Schrödinger equation, blow up criteria, scattering
PACS: 02.30.Jr, 05.30.Jp, 52.38.Hb

The focusing nonlinear Schrödinger equation (NLS) or Gross-Pitaevskii equation with cubic nonlinearity is

$$i\partial_t u + \Delta u + |u|^2 u = 0\,, \tag{1}$$

with the wave function $u = u(x,t) \in \mathbb{C}$. We consider $x \in \mathbb{R}^3$ as this equation arises in Langmuir turbulence in a weakly magnetized plasma (as a limit of the Zakharov system when sending the wave speed to infinity) and in Bose-Einstein condensate with zero (external) potential. The initial-value problem is locally well-posed[2] in H^1 by the classical results of Ginibre and Velo. In this now standard theory obtained from the Strichartz estimates, initial data $u_0 \in H^1$ give rise to a unique solution $u(t) \in C([0,T];H^1)$ with the time interval $[0,T]$ of existence specified in terms of $\|\nabla u_0\|_{L^2}$. In some situations, an *a priori* bound on $\|\nabla u(t)\|_{L^2}$ can be deduced from conservation laws which implies that the solution $u(t)$ exists globally in time. On the other hand, we say that a solution $u(t)$ to NLS *blows-up in finite time T* provided

$$\lim_{t \nearrow T} \|\nabla u(t)\|_{L^2} = +\infty\,. \tag{2}$$

On their maximal interval of existence solutions conserve mass $M[u]$, momentum $P[u]$, and energy $E[u]$, where

$$M[u] = \|u\|_{L^2}^2, \quad P[u] = \mathrm{Im}\int \bar{u}\nabla u\,, \quad E[u] = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{4}\|u\|_{L^4}^4\,.$$

Also, NLS satisfies the scaling symmetry: if $u(x,t)$ is a solution of NLS, then so is $\lambda u(\lambda x, \lambda^2 t)$. Consequently, the scale-invariant Sobolev and Lebesque spaces are $\dot{H}^{1/2}(\mathbb{R}^3)$ and $L^3(\mathbb{R}^3)$. Let

$$V[u](t) = \|xu(t)\|_{L_x^2}^2$$

denote the variance. Assuming $V[u](0) < \infty$, the virial identities (Vlasov-Petrishchev-Talanov [10], Zakharov [11], Glassey [4])

$$\partial_t V[u] = 4\mathrm{Im}\int (x \cdot \nabla u)\,\bar{u}\,dx \quad \text{and} \quad \partial_t^2 V[u] = 24E[u] - 4\|\nabla u\|_{L_x^2}^2 \tag{3}$$

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[1] Extended abstract for ICNAAM 2009, Date: 1 August 2009
[2] This means local existence, uniqueness and continuous dependance on the initial datum.

hold and provide existence of finite time blow up solutions, for example, when $E < 0$. If $Q = Q(x)$ denotes the real-valued, smooth, exponentially decaying ground state solution to

$$-Q+\Delta Q+Q^3=0, \tag{4}$$

then $u(x,t) = e^{it}Q(x)$ solves NLS, and is called the *ground state soliton*.

Global behavior of solutions such as scattering and blow-up criteria are most naturally expressed in terms of scale invariant quantities, and natural candidates are the L^3 norm and the $\dot{H}^{1/2}$ norm. It turns out that the L^3 norm is inadequate, and while the $\dot{H}^{1/2}$ norm is a more reasonable choice, it too appears deficient. In [5, 6, 1], we work instead with two scale-invariant quantities: the mass-energy $M[u]E[u]$ and the (rescaled) gradient

$$\eta(t) \stackrel{\text{def}}{=} \frac{\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}}{\|Q\|_{L^2}\|\nabla Q\|_{L^2}}. \tag{5}$$

The first results classifies the behavior of solutions “under the mass-energy" threshold:

Theorem 1 (Duyckaerts-Holmer-Roudenko [1], Holmer-Roudenko [5, 6, 7])**.** *Suppose that $u_0 \in H^1$ and $M[u]E[u] < M[Q]E[Q]$.*

(1) If $\eta(0) < 1$, then $u(t)$ is globally well-posed[3] and, in fact, scatters[4] in both time directions.

(2a) If $\eta(0) > 1$ and either u_0 has finite variance or u_0 is radial, then $u(t)$ blows-up in finite positive time and finite negative time.

(2b) If $\eta(0) > 1$, then either $u(t)$ blows-up in finite forward time or there exists a sequence $t_n \nearrow +\infty$ such that $\|\nabla u(t_n)\|_{L^2} = \infty$ (so called weak blow up). A similar statement holds for negative time.

The blow up for finite variance as in (2a) has previously been obtained in [9].

The next result describes the behavior of solutions “at the mass-energy" threshold. First part shows the existence of special solutions $Q^{\pm}$, the second part classifies all the solutions at this threshold.

Theorem 2 (Duyckaerts-Roudenko [2])**.** *I. There exist two radial solutions $Q^{\pm}$ of NLS ($Q_0^{\pm} \in \cap_{s>0}H^s(\mathbb{R}^3)$) with $M[Q^{\pm}] = M[Q]$, $E[Q^{\pm}] = E[Q]$ and initial conditions $\|\nabla Q^{-}(0)\|_2 < \|\nabla Q\|_2, \|\nabla Q^{+}(0)\|_2 > \|\nabla Q\|_2$ such that*

(a) $\forall t \geq 0, \quad \left\|Q^{\pm}(t) - e^{it}Q\right\|_{H^1} \leq Ce^{-e_0 t}$, where $-e_0$ is the negative eigenvalue of the linearized operator,

(b) Q^{-} is globally defined and scatters for negative time,

(c) the negative time of existence of Q^{+} is finite.

II. Let u be a solution of NLS satisfying $M[u]E[u] = M[Q]E[Q]$.

(a) If $\eta(0) < 1$, then either u scatters or $u = Q^{-}$ (up to the equation symmetries).

(b) If $\eta(0) = 1$, then $u = e^{it}Q$ (up to the symmetries).

(c) If $\eta(0) > 1$, and u_0 is radial or of finite variance, then either the interval of existence of u is of finite length (and thus, u blows up) or $u = Q^{+}$ (up to the symmetries).

Next we discuss some alternate criteria for blow-up in the spirit of Lushnikov [8]. These criteria, for instance, show the existence of blow up solutions “above the mass-energy" threshold. First, we state his result, adapted to our notation; we restrict to the case $E > 0$, since $E \leq 0$ is comparatively well understood from Theorem 1.

Theorem 3 (adapted from Lushnikov [8])**.** *Suppose that $u_0 \in H^1$ and $\|xu_0\|_{L^2} < \infty$. The following is a sufficient condition for blow-up in finite time:*

$$\frac{V_t(0)}{M} < 2\sqrt{3}\, g\left(\frac{8EV(0)}{3M^2}\right), \quad \textit{where} \quad g(\omega) = \begin{cases} \sqrt{\frac{2}{\omega^{1/2}}+\omega-3} & \textit{if} \quad 0<\omega\leq 1 \\ -\sqrt{\frac{2}{\omega^{1/2}}+\omega-3} & \textit{if} \quad \omega \geq 1. \end{cases} \tag{6}$$

For the real-valued initial data Theorem 3 states the blow up criterion as

$$V(0) < \frac{3}{8}\frac{M^2}{E}. \tag{7}$$

[3] The local solution is unique, can be extended globally in time and there is continuous dependence on the initial data.

[4] By scattering here we mean the obtained solution approaches the solution of the linear equation as time goes to $\pm$ infinity.

Theorem 3 is based upon use of the uncertainty principle

$$\|u\|_{L^2}^4 + \frac{4}{9}\left|\mathrm{Im}\int (x\cdot\nabla u)\bar{u}\,dx\right|^2 \le \frac{4}{9}\,\|xu\|_{L^2}^2\,\|\nabla u\|_{L^2}^2\,, \tag{8}$$

the virial identity, and a "mechanical analysis" of the resulting second-order ODE in $V(t)$. By replacing (8) with

$$\|u\|_{L^2} \le C\,\|xu\|_{L^2}^{3/7}\|u\|_{L^4}^{4/7} \quad \text{with the sharp constant} \quad C = \left(\frac{2^2\cdot 7^5\cdot\pi^2}{3^5\cdot 5^2}\right)^{1/14}, \tag{9}$$

we obtain a different condition which in some cases improves upon Theorem 3. The inequality (9) can be thought of as a variant of the Hölder interpolation inequality $\|u\|_{L^2} \le \|u\|_{L^{6/5}}^{3/7}\|u\|_{L^4}^{4/7}$, since $\|u\|_{L^{6/5}}$ and $\|xu\|_{L^2}$ scale the same way.

Theorem 4. *Suppose that $u_0 \in H^1$ and $\|xu_0\|_{L^2} < \infty$. The following is a sufficient condition for blow-up in finite time:*

$$\frac{V_t(0)}{M} < \frac{2\sqrt{2}\,(ME)^{1/6}}{C^{7/3}}\; g\left(4\left(\frac{C^{14}E^2}{M^7}\right)^{1/3} V(0)\right) \quad \textit{with} \quad g(x) \quad \textit{from} \quad (6). \tag{10}$$

For real-valued data the sufficient condition for blow up from Theorem 4 is $V(0) < c\,\dfrac{M^{7/3}}{E^{2/3}}$ with $c = \left(\dfrac{3^5 5^2}{2^8 7^5\pi^2}\right)^{1/3}$, which is an improvement over (7) when $M[u]E[u] > \frac{7^5\pi^2}{2\cdot 3^2 5^2} \approx 2.06 M[Q]E[Q]$ (for real-valued data).

To illustrate Theorems 3 and 4, we consider two examples of initial data: (1) a Q profile with a quadratic phase and (2) a super gaussian.

Example 5. *Consider $u_0 = \lambda^{3/2}Q(\lambda r)\,e^{i\gamma r^2}$, $\gamma < 0$, and note that $M[u_0]E[u_0] = \left(1+4\gamma^2\frac{\|yQ\|_2^2}{\|Q\|_2^2}\right)M[Q]E[Q] > M[Q]E[Q]$, thus, Theorems 1 or 2 are not applicable. In Figure 1 we plot all conditions from Theorems 1-4. In a special case with $\lambda = 1$, Theorem 3 implies that such solution will blow up in finite time if*

$$\gamma < -\left(\frac{3}{4}\,\frac{\|Q\|_2^4}{\|yQ\|_2^4\left(3-\frac{4}{3}\frac{\|Q\|_2^2}{\|yQ\|_2^2}\right)^2} - \frac{\|Q\|_2^2}{4\|yQ\|_2^2}\right)^{1/2} \approx -0.177,$$

or by Theorem 4 if

$$\gamma < -\left(\frac{\|Q\|_2^4}{54C^7\|yQ\|_2^2}\left(\frac{\|Q\|_2}{\|yQ\|_2} + \frac{2C^7\,\|yQ\|_2^2}{\|Q\|_2^4}\right)^3 - \frac{\|Q\|_2^2}{4\,\|yQ\|_2^2}\right)^{1/2} \approx -0.279.$$

In this case Theorem 3 is more powerful than Theorem 4, however, this is not always the case, as is shown in Figure 1.

Example 6. *Let $u_0(r) = p e^{-\alpha r^4/2}$. In Figure 2 we illustrate conditions from Theorems 1-4, where for clarity and scaling considerations, the axes are chosen $p/\sqrt[4]{\alpha}$ versus α. Theorem 4 provides the best theoretical bound for the blow up in finite time and Theorem 1 (part 1) provides the best bound for the scattering. It is conjectured that the threshold for this kind of initial data (or more generally, real-valued initial data) is provided by the $\|Q\|_{\dot{H}^{1/2}}$ as it can be seen from the numerical calculations presented by a dotted line in Figure 1. However, for nonreal initial data we have numerical evidence that this condition does not provide a scattering threshold.*

Finally, we give a radial, infinite-variance version of Theorem 4 based upon a local virial identity and a bootstrap argument. Select a smooth radial (nonstrictly) increasing function $\psi(x)$ such that $\psi(x) = |x|^2$ for $0 \le |x| \le 1$ and $\psi(x) = 2$ for $|x| \ge 2$. Define the *localized variance*

$$V_R \stackrel{\text{def}}{=} \int R^2\psi\left(\frac{x}{R}\right)|u(x)|^2\,dx. \tag{11}$$

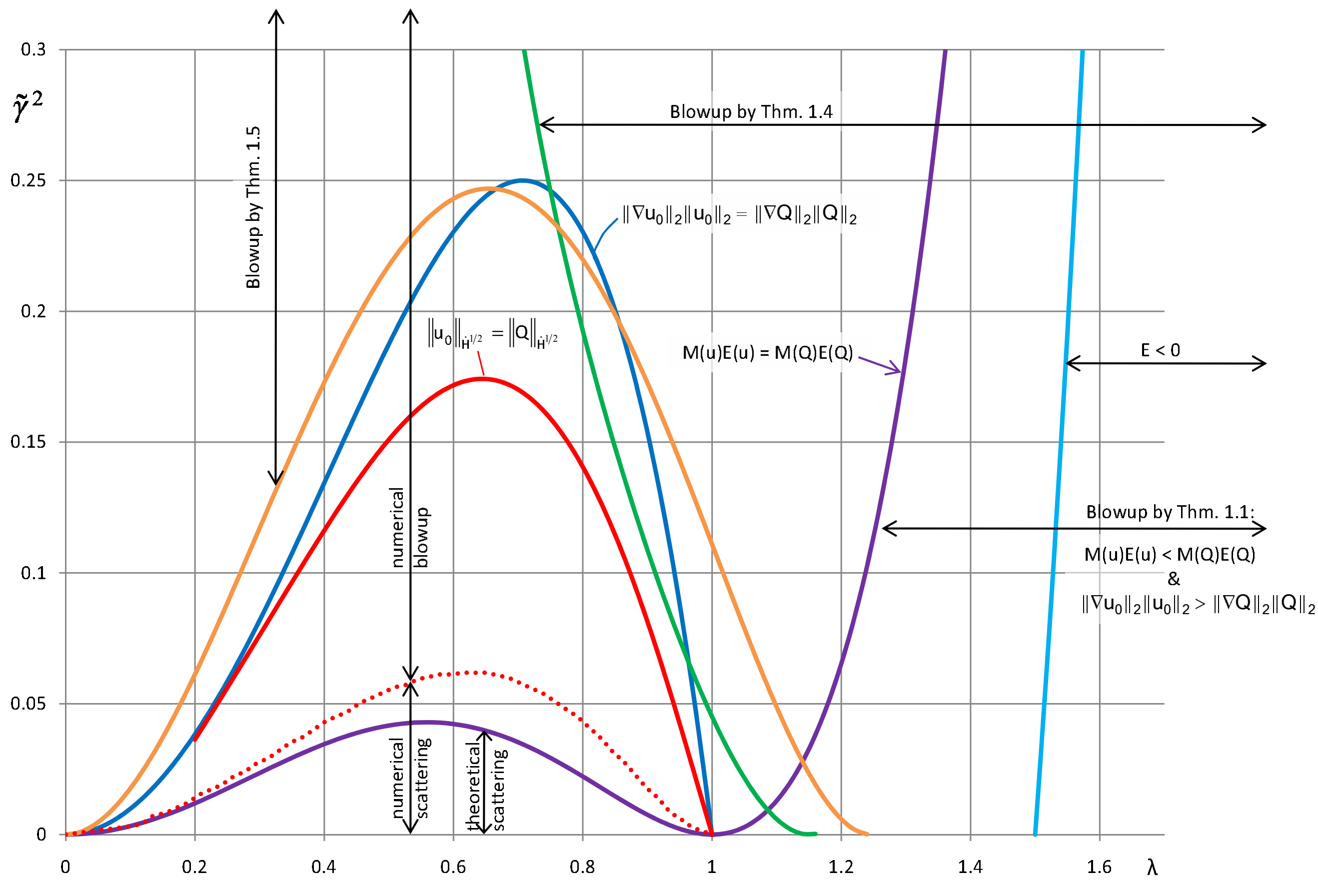

FIGURE 1. Global behavior of the solutions to the Q profile initial data with the negative quadratic phase $u_0(r) = \lambda^{3/2}Q(\lambda r)\,e^{-i|\gamma|r^2}$. Here, $\tilde{\gamma}$ is the renormalized γ: $\tilde{\gamma}^2 = \frac{4}{3}\,\|yQ\|_2^2/\|Q\|_2^2\gamma^2 \approx 1.43\gamma^2$.

Theorem 7. *Fix $\delta \ll 1$ and suppose $ME > 1$. Given $u_0 \in H^1$, take any R such that $\frac{V_R(0)}{M} \leq \frac{1}{2}R^2, R^2 \gtrsim \frac{M^2}{\delta}$. Then the following is a sufficient condition for blow-up in finite time:*

$$\frac{(V_R)_t(0)}{M} < \sqrt{6}\,(8+\delta)^{\frac{1}{6}}\,\tilde{C}^{\frac{7}{3}}\,(ME)^{\frac{1}{6}}\,g\left(\frac{(8+\delta)^{\frac{2}{3}}\tilde{C}^{\frac{14}{3}}E^{\frac{2}{3}}}{M^{\frac{7}{3}}}V_R(0)\right), \quad \tilde{C} = \left(\frac{2^{11}\pi^2}{3^2}\right)^{\frac{1}{14}},$$

where g is defined in (6).

Note that by the dominated convergence theorem, for any $u_0 \in L^2$, we have $\lim_{R\to+\infty}\frac{V_R(0)}{R^2M} = 0$. Thus, there always exists R such that the assumption of the Theorem 7 holds.

Results for NLS equations in other dimensions d and nonlinearities p (when the critical index $0 < s = \frac{d}{2} - \frac{2}{p-1} < 1$) are obtained, for example, in [5] and [3].

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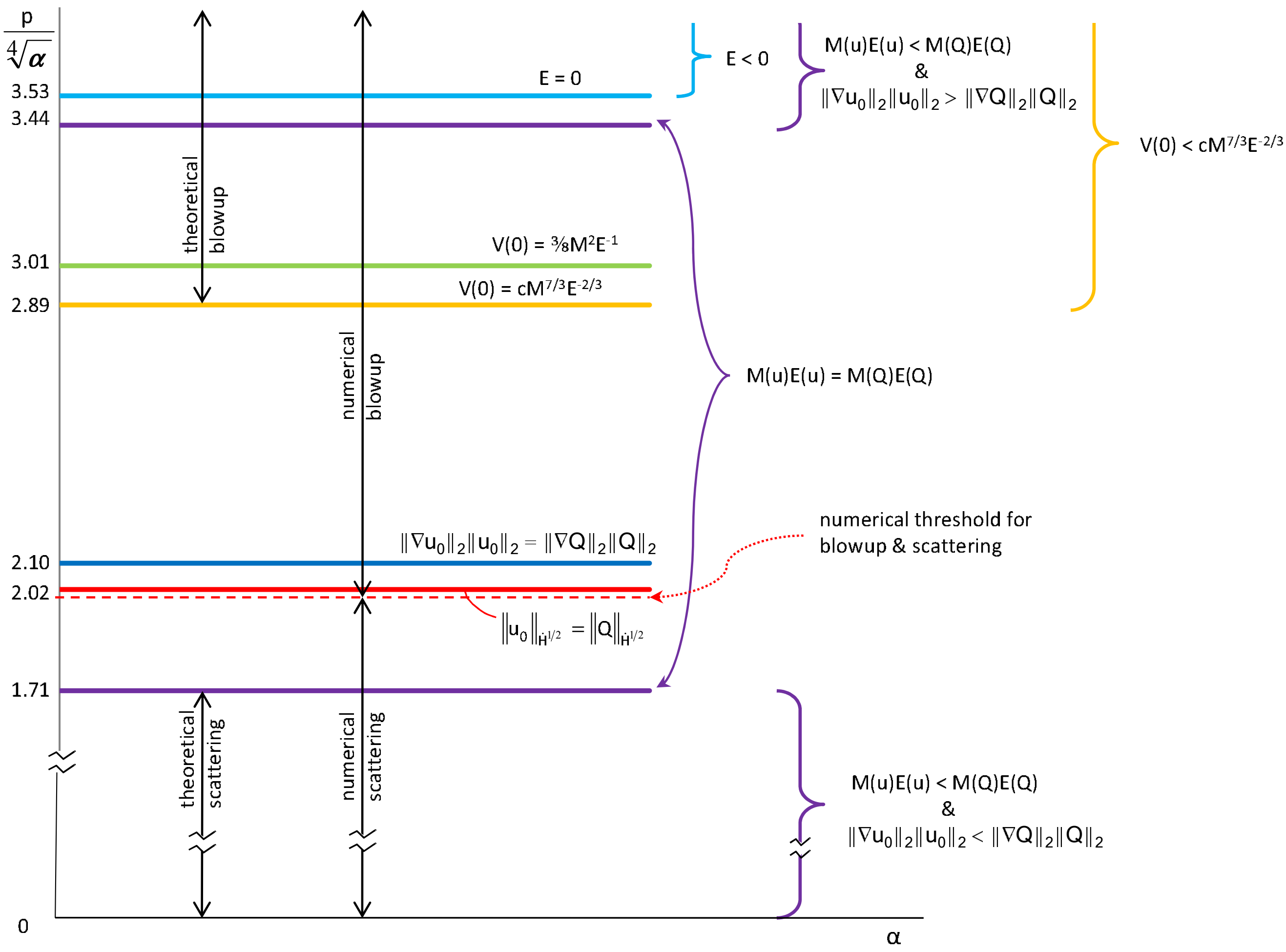

FIGURE 2. Global behavior of the solutions with the super gaussian initial data $u_0(r) = pe^{-\alpha r^4/2}$.

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