Reference number: SM 4991

DUALITY OF MATRIX-WEIGHTED BESOV SPACES

SVETLANA ROUDENKO

Abstract. We determine the duals of the homogeneous matrix-weighted Besov spaces $\dot{B}_p^{\alpha q}(W)$ and $\dot{b}_p^{\alpha q}(W)$ which were previously defined in [5]. If W is a matrix $A_p$weight, then the dual of $\dot{B}_p^{\alpha q}(W)$ can be identified with $\dot{B}_{p'}^{-\alpha q'}(W^{-p'/p})$ and, similarly, $[\dot{b}_p^{\alpha q}(W)]^* \approx \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p})$. Moreover, for certain W which may not be in the A_p class, the duals of $\dot{B}_p^{\alpha q}(W)$ and $\dot{b}_p^{\alpha q}(W)$ are determined and expressed in terms of the Besov spaces $\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ and $\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$, which we define in terms of reducing operators $\{A_Q\}_Q$ associated with W. We also develop the basic theory of these reducing operator Besov spaces. Similar results are shown for inhomogeneous spaces.

Contents

1. Introduction 1
2. Definitions and Notation 5
3. Duality of sequence Besov spaces 7
4. Equivalence of sequence and discrete averaging Besov spaces 11
5. Properties of averaging L^p spaces 15
6. Duality of continuous Besov spaces 19
7. Duality of inhomogeneous Besov spaces $b_p^{\alpha q}(W)$ and $B_p^{\alpha q}(W)$ 23
8. Acknowledgements 25
References 25

1. Introduction

The aim of this paper is to determine the duals of the Besov function spaces $\dot{B}_p^{\alpha q}(W)$ and the corresponding sequence spaces $\dot{b}_p^{\alpha q}(W)$ for $\alpha \in \mathbb{R}$, $0 < q < \infty$ and $1 < p < \infty$. Here, W is a matrix weight, namely, an a.e. invertible map from $\mathbb{R}^n$ to the cone $\mathcal{M}$ of non-negative definite operators on a Hilbert space $\mathcal{H}$ of dimension m (e.g. $\mathcal{H} = \mathbb{C}^m$ or $\mathbb{R}^m$), i.e., for a.e. $t \in \mathbb{R}^n$, $(W(t)x, x)_{\mathcal{H}} \geq 0$ for all $x \in \mathcal{H}$.

To understand what properties of W are needed to identify dual spaces, we will heavily use the technique of *reducing operators* (for definitions refer to Section 2 or [5], [10]). Namely, instead of dealing with matrix weights, we consider a sequence of matrices enumerated by dyadic cubes and establish properties of Besov spaces with

---

2000 *Mathematics Subject Classification.* Primary 42B25, 42B35, 46A20, 46B10. Secondary 47B38.

Key words and phrases. Duality, Besov spaces, matrix weights, reducing operators, A_p condition, doubling measure, φ-transform.

such sequences of matrix weights. Then, given a matrix W, its reducing operators constitute such a sequence.

Denote by $\mathcal{D}$ the collection of dyadic cubes in $\mathbb{R}^n$ and for each $Q \in \mathcal{D}$ let A_Q be a positive-definite (thus, self-adjoint) operator on $\mathcal{H}$. Also denote by $\mathcal{RS}_{\mathcal{D}}$ (reducing sequences) the collection of all sequences $\{A_Q\}_{Q\in\mathcal{D}}$ of positive-definite operators on $\mathcal{H}$. An admissible kernel $\varphi \in \mathcal{A}$ is a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$such that supp $\hat{\varphi} \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\}$ and $|\hat{\varphi}(\xi)| \ge c > 0$ if $\frac{3}{5} \le |\xi| \le \frac{5}{3}$. Set $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$ for $\nu \in \mathbb{Z}$.

In [5] we introduced the following:

Definition 1.1 (*Matrix-weighted Besov space* $\dot{B}_p^{\alpha q}(W)$). For $\alpha \in \mathbb{R}$, $1 \le p < \infty$, $0 < q \le \infty$, $\varphi \in \mathcal{A}$ and W a matrix weight, the Besov space $\dot{B}_p^{\alpha q}(W)$ is the collection of all vector-valued distributions $\vec{f} = (f_1, ..., f_m)^T$ with $f_i \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$ (the space of tempered distributions modulo polynomials), $1 \le i \le m$ such that

$$\left\|\vec{f}\right\|_{\dot{B}_p^{\alpha q}(W)} = \left\|\left\{2^{\nu\alpha}\|\varphi_\nu * \vec{f}\,\|_{L^p(W)}\right\}_\nu\right\|_{l^q} = \left\|\left\{\left\|W^{1/p}(t)\cdot(\varphi_\nu * \vec{f}\,)(t)\right\|_{L^p(dt)}\right\}_\nu\right\|_{l_q^\alpha} < \infty,$$

where $\varphi_\nu * \vec{f} = (\varphi_\nu * f_1, ..., \varphi_\nu * f_m)^T$.

Suppose W satisfies **any** of the three conditions:

(A1) $W \in A_p$ with $1 < p < \infty$,
(A2) W is a doubling matrix of order p with $p > \beta$, where β is the doubling exponent of W,
(A3) W is a diagonal doubling matrix of order p with $1 \le p < \infty$.

(For definitions refer to Section 2.) Then $\dot{B}_p^{\alpha q}(W)$ is independent of the choice of $\varphi \in \mathcal{A}$ ([5, Theorem 1.8]). If a matrix weight W satisfies none of (A1)-(A3), then there may be a dependence on φ (i.e., $\dot{B}_p^{\alpha q}(W, \varphi)$), nevertheless, all results will hold up to a choice of an admissible kernel φ.

Here, as a main tool and a useful object by itself, we define the space $\dot{B}_p^{\alpha q}(\{A_Q\})$ with a sequence of discrete weights $\{A_Q\}_Q$:

Definition 1.2 (*Averaging matrix-weighted Besov space* $\dot{B}_p^{\alpha q}(\{A_Q\})$). For $\alpha \in \mathbb{R}$, $1 \le p \le \infty$, $0 < q \le \infty$, $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$ and $\varphi \in \mathcal{A}$,the Besov space $\dot{B}_p^{\alpha q}(\{A_Q\})$ is the collection of all vector-valued distributions$\vec{f} = (f_1, ..., f_m)^T$ with $f_i \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$, $1 \le i \le m$ such that

$$\left\|\vec{f}\right\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} = \left\|\left\{2^{\nu\alpha}\left\|\sum_{l(Q)=2^{-\nu}}\left\|A_Q\cdot(\varphi_\nu * \vec{f}\,)\right\|_{\mathcal{H}}\chi_Q\right\|_{L^p}\right\}_\nu\right\|_{l^q} < \infty,$$

where $l(Q)$ is the side length of Q.

This space is well-defined (i.e., independent of $\varphi \in \mathcal{A}$), see Corollary 4.9, if $\{A_Q\}_Q$ is a doubling matrix sequence defined as follows.

Definition 1.3 (*Doubling sequence*). We say $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$ is a *(dyadic) doubling sequence (of order* p, $1 \le p < \infty$*)*, if there exists $\beta \ge n$ and $c \ge 1$ such that for all

P, Q dyadic

$$\|A_Q A_P^{-1}\|^p \le c\,\frac{|P|}{|Q|}\,\max\left(1, \left[\frac{l(Q)}{l(P)}\right]^{\beta}\right)\left(1+\frac{\operatorname{dist}(P,Q)}{\max(l(P),l(Q))}\right)^{\beta}, \tag{1}$$

where $|Q|$ is the Lebesgue measure of Q and the norm on the left side is the operator (matrix) norm.

Observe that if (1) holds for some p, then it holds for $1 \le q < p$, since the right-hand side is ≥ 1.

Our main goal is to identify the dual space of $\dot{B}_p^{\alpha q}(W)$. For $W \in A_p$ the result can be expressed in terms of matrix weights. However, even for $W \notin A_p$ but satisfying (A2) or (A3), we are able to characterize $\left[\dot{B}_p^{\alpha q}(W)\right]^*$ in terms of reducing operators. Set $\frac{1}{p}+\frac{1}{p'}=1$ if $1<p<\infty$ and $p'=\infty$ if $p=1$; $\frac{1}{q}+\frac{1}{q'}=1$ if $1<q<\infty$ and $q'=\infty$ if $0<q\le 1$. It is important to emphasize our convention for the duality pairing. In what follows, we say that a function space Y is a dual of a function space X, $X^* \approx Y$, in the sense that each $y \in Y$ defines an element l_y of X^* via the pairing $l_y(x) = (x,y) = \int_{R^n} \langle x(t), y(t)\rangle\, dt$ and every element of X^* is of the kind l_y for some $y \in Y$ with $\|l_y\| \approx \|y\|_Y$. (For example, $[L^p(W)]^* \approx L^{p'}(W^{-p'/p})$, $1<p<\infty$, with the pairing $(\vec{f},\vec{g}) = \int_{R^n} \left\langle \vec{f}(t), \vec{g}(t)\right\rangle_{\mathcal{H}} dt$; refer to Section 3 for more details.)

Theorem A 1. *Let $\alpha \in \mathbb{R}$, $1 \le p < \infty$, $0 < q < \infty$ and let $\{A_Q\}_Q$ be reducing operators of a matrix weight W.*

$$\text{If } W \in A_p, 1<p<\infty, \text{ then } \left[\dot{B}_p^{\alpha q}(W)\right]^* \approx \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}). \tag{2}$$

$$\text{If } W \text{ satisfies any of (A1)-(A3), then } \left[\dot{B}_p^{\alpha q}(W)\right]^* \approx \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}). \tag{3}$$

(For the proof refer to Section 5.)

Our next result identifies the dual space of the sequence (discrete) Besov space $\dot{b}_p^{\alpha q}(W)$. The connection between $\dot{b}_p^{\alpha q}(W)$ and $\dot{B}_p^{\alpha q}(W)$ is that $\vec{f} \in \dot{B}_p^{\alpha q}(W)$ if and only if the appropriate wavelet coefficient sequence of $\vec{f}$ belongs to $\dot{b}_p^{\alpha q}(W)$ (see [5] for details). Recall the definitions of $\dot{b}_p^{\alpha q}(W)$ and $\dot{b}_p^{\alpha q}(\{A_Q\})$ from [5]:

Definition 1.4 (*Matrix-weighted sequence Besov space $\dot{b}_p^{\alpha q}(W)$*)**.** For $\alpha \in \mathbb{R}$, $1 \le p < \infty$, $0 < q \le \infty$ and W a matrix weight, the space $\dot{b}_p^{\alpha q}(W)$ consists of all vector-valued sequences $\vec{s} = \{\vec{s}_Q\}_{Q\in\mathcal{D}}$, where $\vec{s}_Q = \left(s_Q^{(1)}, ..., s_Q^{(m)}\right)^T$, such that

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} = \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \left(\|W^{1/p}(t)\vec{s}_Q\|_{\mathcal{H}}\right) \chi_Q(t) \right\|_{L^p(dt)} \right\}_{\nu} \right\|_{l^q} < \infty.$$

Definition 1.5 (*Averaging matrix-weighted discrete Besov space $\dot{b}_p^{\alpha q}(\{A_Q\})$*)**.** For $\alpha \in \mathbb{R}$, $1 \le p \le \infty$, $0 < q \le \infty$ and $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$, the space $\dot{b}_p^{\alpha q}(\{A_Q\})$ consists of all vector-valued sequences $\{\vec{s}_Q\}_{Q\in\mathcal{D}}$ such that

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} = \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \left(\|A_Q\vec{s}_Q\|_{\mathcal{H}}\right) \chi_Q(t) \right\|_{L^p(dt)} \right\}_\nu \right\|_{l^q}$$
$$= \left\| \{A_Q\vec{s}_Q\}_Q \right\|_{\dot{b}_p^{\alpha q}} < \infty.$$

If $\{A_Q\}_Q$ is a sequence of reducing operators for a matrix weight W, then the norm equivalence

$$\|\vec{s}\|_{\dot{b}_p^{\alpha q}(W)} \approx \|\vec{s}\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} \tag{4}$$

holds for any matrix weight W, $\alpha \in \mathbb{R}$, $0 < q \le \infty$ and $1 \le p < \infty$ ([5, Lemma 7.1]). To make notation short, we will write $\dot{b}_p^{\alpha q}(W) \approx \dot{b}_p^{\alpha q}(\{A_Q\})$ for the norm equivalence.

Theorem A 2. *Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \le p < \infty$ and let $\{A_Q\}_Q$ be reducing operators of a matrix weight W. Then*

$$\left[\dot{b}_p^{\alpha q}(W)\right]^* \approx \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}). \tag{5}$$

Moreover, if $W \in A_p$, $1 < p < \infty$, then

$$\left[\dot{b}_p^{\alpha q}(W)\right]^* \approx \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p}). \tag{6}$$

The paper is organized as follows. In Section 3 we discuss the discrete Besov space $\dot{b}_p^{\alpha q}(W)$. We use a “one at a time reduction” approach meaning we reduce the space $\dot{b}_p^{\alpha q}(W)$ in the following order:

$$\dot{b}_p^{\alpha q}(W) \longrightarrow \dot{b}_p^{\alpha q}(\{A_Q\}) \longrightarrow \dot{b}_p^{\alpha q}(\mathbb{R}^m) \longrightarrow \dot{b}_p^{\alpha q}(\mathbb{R}^1),$$

where the last two spaces are unweighted vector-valued and scalar-valued discrete Besov spaces, and then identify the duals in the opposite order. A similar approach is used for $\dot{B}_p^{\alpha q}(W)$.

The fact that each A_Q is constant on each dyadic cube Q allows us establish

$$\left[\dot{b}_p^{\alpha q}(\{A_Q\})\right]^* \approx \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \tag{7}$$

for any $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$, $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \le p < \infty$. If the A_Q's are generated by a matrix weight W, then combining (4) and (7), we get (5) of Theorem A 2.

In order to connect $\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ with $\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{\#}\}) \approx \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p})$ (for the definition of $A_Q^{\#}$ refer to Section 2) the matrix A_p condition is needed, though only for one direction of the embedding; the other direction is automatic. Thus, the following chain of the equivalences holds for $\dot{b}_p^{\alpha q}(W)$:

$$\left[\dot{b}_p^{\alpha q}(W)\right]^* \overset{\text{any } W}{\approx} \left[\dot{b}_p^{\alpha q}(\{A_Q\})\right]^* \approx \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \overset{A_p}{\approx} \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{\#}\}) \overset{\text{any } W}{\approx} \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p}). \tag{8}$$

This completes the proof of Theorem A 2.

In Section 4 we prove the norm equivalence between $\dot{B}_p^{\alpha q}(\{A_Q\})$ and $\dot{b}_p^{\alpha q}(\{A_Q\})$ for any doubling sequence $\{A_Q\}_Q$. Note that if the A_Q's are generated by a matrix weight

W, then all that is required from the weight is the doubling condition. (Compare this with (A1)-(A3) conditions from [5] for the norm equivalence between the original matrix-weighted Besov spaces.)

For $Q = Q_{\nu k} = \prod_{i=1}^{n} \left[\frac{k_i}{2^\nu}, \frac{k_i+1}{2^\nu}\right]$, $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, denote $\varphi_Q(x) = |Q|^{-1/2}\varphi(2^\nu x - k) = |Q|^{1/2}\varphi_\nu(x - x_Q)$, where $x_Q = 2^{-\nu}k$ is the lower left corner of $Q_{\nu k}$.

Theorem 1.6. *Let $\alpha \in \mathbb{R}$, $0 < q \le \infty$, $1 \le p < \infty$ and $\{A_Q\}_Q$ be a doubling sequence (of order p). Then for $\vec{s}_Q(\vec{f}) = \left\langle \vec{f}, \varphi_Q \right\rangle$*

$$\left\|\vec{f}\right\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} \approx \left\|\left\{\vec{s}_Q\left(\vec{f}\right)\right\}_Q\right\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}.$$

In Section 5 we establish the correspondence between the continuous Besov spaces $\dot{B}_p^{\alpha q}(W)$ and $\dot{B}_p^{\alpha q}(\{A_Q\})$.

Lemma 1.7. *Let $\alpha \in \mathbb{R}$, $0 < q \le \infty$ and $1 \le p < \infty$. If W satisfies any of (A1)-(A3) and $\{A_Q\}_Q$ is a sequence of reducing operators generated by W, then*

$$\dot{B}_p^{\alpha q}(W) \approx \dot{B}_p^{\alpha q}(\{A_Q\}).$$

For one direction of the above equivalence it suffices to have W doubling.

In Section 6 it is shown that if $\{A_Q\}_Q$ is a doubling sequence of order p, $1 \le p < \infty$, then

$$\left[\dot{B}_p^{\alpha q}(\{A_Q\})\right]^* \approx \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}). \tag{9}$$

Using the above duality and equivalence, we get the following chain:

$$[\dot{B}_p^{\alpha q}(W)]^* \overset{(1)}{\approx} [\dot{B}_p^{\alpha q}(\{A_Q\})]^* \approx \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \overset{A_p}{\approx} \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{\#}\}) \overset{(4)}{\approx} \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}), \tag{10}$$

where the equivalence (1) holds if W satisfies any of (A1)-(A3) and (4) holds if $W^{-p'/p}$ satisfies any of (A1)-(A3) properly adjusted (see section 2). The third equivalence holds under the A_p condition. However, the A_p condition is needed only for one direction of the embedding. This proves Theorem A 1.

In the last section we consider inhomogeneous function spaces and transfer all the above theory to the inhomogeneous case.

2. Definitions and Notation

Given a matrix weight W, for each dyadic cube Q in $\mathbb{R}^n$ consider a *reducing operator* A_Q corresponding to the L^p average over Q of the norm $\|W^{1/p} \cdot \|_{\mathcal{H}}$, i.e.,

$$\|A_Q\, x\|_{\mathcal{H}} \approx \left(\frac{1}{|Q|}\int_Q \|W^{1/p}(t)\, x\|_{\mathcal{H}}^p\, dt\right)^{1/p} \quad \text{for all } x \in \mathcal{H}.$$

Thus, we are dealing with a family of norms $\rho_t(x) = \|W^{1/p}(t)\, x\|_{\mathcal{H}}$. By definiton, the dual norms are $\rho_t^*(x) = \|W^{-1/p}(t)\, x\|_{\mathcal{H}}$ and reducing operators for their $L^{p'}$ averages over a cube Q are

$$\|A_Q^{\#}\, x\|_{\mathcal{H}} \approx \left(\frac{1}{|Q|} \int_Q \|W^{-1/p}(t)\, x\|_{\mathcal{H}}^{p'}\, dt \right)^{1/p'} .$$

In other words, $\{A_Q^{\#}\}_Q$ is a reducing sequence for the "dual" matrix weight $W^{-p'/p}$ (for more details refer to [10], [5]).

Recall that the matrix A_p condition is $\|A_Q^{\#}\, A_Q\| \leq c$ for every cube $Q \subseteq \mathbb{R}^n$, and the opposite inequality $\|(A_Q^{\#}\, A_Q)^{-1}\| \leq c$ always holds as a simple consequence of Hölder's inequality: for any $x,\, y \in \mathcal{H}$ we have

$$|(x,y)| \leq \left(\int_Q \|W^{1/p}(t)\, x\|^p \frac{dt}{|Q|} \right)^{1/p} \left(\int_Q \|W^{-1/p}(t)\, y\|^{p'} \frac{dt}{|Q|} \right)^{1/p'} \approx \|A_Q\, x\|\, \|A_Q^{\#}\, y\|,$$

which implies $\|A_Q\, x\| \geq c\, \|(A_Q^{\#})^{-1}\, x\|$ for any $x \in \mathcal{H}$ and, thus, the above statement follows.

A condition which is weaker than A_p for a matrix weight W is the doubling condition:

Definition 2.1 (*Doubling matrix*)**.** A matrix weight $W : \mathbb{R}^n \to \mathcal{M}$ is called a *doubling matrix (of order p, $1 \leq p < \infty$)*, if there exists a constant $c = c_{p,n}$ such that for any $x \in \mathcal{H}$, any $\delta > 0$ and any $z \in \mathbb{R}^n$

$$\int_{B_{2\delta}(z)} \|W^{1/p}(t)\, x\|_{\mathcal{H}}^p\, dt \leq c \int_{B_\delta(z)} \|W^{1/p}(t)\, x\|_{\mathcal{H}}^p\, dt, \tag{11}$$

i.e., the scalar measure $w_x(t) = \|W^{1/p}(t)\, x\|_{\mathcal{H}}^p$ is uniformly doubling and not identically zero (a.e.). If $c = 2^\beta$ is the smallest constant for which (11) holds, then β is called the *doubling exponent*of W.

Observe that if W is a doubling matrix weight (of order p), then $\{A_Q\}_Q$ is a doubling sequence (of order p). The fact that A_p implies doubling in the scalar case is quite straightforward and can be found in [7]. The vector case can be found in [6]. Also note that $\beta \geq n$ and for the Lebesgue measure $\beta = n$.

By saying $W^{-p'/p}$ satisfies any of (A1)-(A3), we mean either $W^{-p'/p} \in A_{p'}$ with $1 < p' < \infty$ (which is equivalent to $W \in A_p$, $1 < p < \infty$), or $W^{-p'/p}$ is a doubling matrix of order p' with $p' > \beta^*$, where β^* is the doubling exponent of $W^{-p'/p}$,or $W^{-p'/p}$ is a diagonal doubling matrix of order p' with $1 < p' < \infty$.

In order to establish the connection between matrix weighted Besov spaces and averaging Besov spaces, we use an auxiliary L^p-space:

Definition 2.2 (*Averaging space $L^p(\{A_Q\}, \nu)$*)**.** For $\nu \in \mathbb{Z}$, $1 \leq p \leq \infty$ and $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$, the space $L^p(\{A_Q\}, \nu)$ consists of all vector-valued locally integrable functions $\vec{f}$ such that

$$\|\vec{f}\,\|_{L^p(\{A_Q\},\nu)} = \left\| \sum_{l(Q)=2^{-\nu}} \chi_Q(t)\, A_Q \vec{f}(t) \right\|_{L^p(dt)} < \infty.$$

Note that $\left\|\vec{f}\right\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} = \left\|\left\{2^{\nu\alpha}\left\|\varphi_\nu * \vec{f}\right\|_{L^p(\{A_Q\},\nu)}\right\}_\nu\right\|_{l^q}$.
To make notation short, define $Q_\nu = \{Q \in \mathcal{D} : l(Q) = 2^{-\nu}\}$.

3. Duality of sequence Besov spaces

An important tool that we need is the duality on $l^q(X)$ with X being a Banach space. By definition $l^q(X)$, $0 < q < \infty$ is the set of all sequences $\{f_\nu\}_{\nu\in\mathbb{Z}}$ with $f_\nu \in X$, $\nu \in \mathbb{Z}$ such that $\left(\sum_{\nu\in\mathbb{Z}} \|f_\nu\|_X^q\right)^{1/q} < \infty$. If $1 \le q < \infty$, then $(l^q(X))^* = l^{q'}(X^*)$ (see [1, Chapter 8]), and if g is a continuous linear functional on $l^q(X)$ identified with $\{g_\nu\}_{\nu\in\mathbb{Z}} \in l^{q'}(X^*)$, then the duality is represented as

$$g(f) = (f, g) = \sum_{\nu\in\mathbb{Z}} \langle f_\nu, g_\nu\rangle_X ,$$

where $\langle f_\nu, g_\nu\rangle_X = g_\nu(f_\nu)$ is the pairing between X and X^*. We will mainly be concerned with $X = L^p$, $1 \le p < \infty$, or $L^p(W)$, $1 < p < \infty$, and, thus, $X^* = L^{p'}$ or $L^{p'}(W^{-p'/p})$, respectively, with the pairing $\langle f, g\rangle_X = \int \langle f(x), g(x)\rangle_{\mathcal{H}}\, dx$.

If $0 < q < 1$, and $X = L^p$, $1 \le p < \infty$, then $(l^q(L^p))^* = l^\infty(L^{p'})$ (see [8, p.177]) and the pairing is defined as above.

Theorem 3.1. *Let W be a matrix weight, $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 < p < \infty$. Then*

(i) $\dot{b}_{p'}^{-\alpha q'}(W^{-p'/p}) \subseteq \left[\dot{b}_p^{\alpha q}(W)\right]^*$ *always*

(ii) $\left[\dot{b}_p^{\alpha q}(W)\right]^* \subseteq \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p})$ *if* $W \in A_p$.

We will prove this theorem, which implies (6) of Theorem A 2, in several steps. The use of reducing operators is essential and helps to understand why certain conditions on the weight W are necessary.

Proof of (i) of Theorem 3.1. For each $\vec{t} \in \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p})$ define a functional $l_{\vec{t}}$ on $\dot{b}_p^{\alpha q}(W)$ by

$$l_{\vec{t}}(\vec{s}) = (\vec{s}, \vec{t}) = \sum_Q \left\langle \vec{s}_Q, \vec{t}_Q\right\rangle_{\mathcal{H}} \text{ for any } \vec{s} = \{\vec{s}_Q\}_Q \in \dot{b}_p^{\alpha q}(W).$$

The calculations below show that this sum converges and $l_{\vec{t}} \in \left[\dot{b}_p^{\alpha q}(W)\right]^*$:

$$\left|\sum_Q \left\langle \vec{s}_Q, \vec{t}_Q\right\rangle_{\mathcal{H}}\right| \le \sum_{\nu\in\mathbb{Z}} \int_{\mathbb{R}^n} \sum_{Q\in Q_\nu} |Q|^{-1} \left|\left\langle W^{-1/p}(x)\, W^{1/p}(x)\vec{s}_Q, \vec{t}_Q\right\rangle_{\mathcal{H}}\right| \chi_Q(x)\, dx. \tag{12}$$

Using the self-adjointness of W and the Cauchy-Schwarz inequality, we bound (12) by

$$\sum_{\nu\in\mathbb{Z}} \int_{\mathbb{R}^n} \sum_{Q\in Q_\nu} \left(|Q|^{-\frac{\alpha}{n}-\frac{1}{2}}\chi_Q(x)\, \|W^{1/p}(x)\vec{s}_Q\|_{\mathcal{H}}\right) \left(|Q|^{\frac{\alpha}{n}-\frac{1}{2}}\chi_Q(x)\, \|W^{-1/p}(x)\vec{t}_Q\|_{\mathcal{H}}\right) dx.$$

Applying Hölder's inequality several times, we estimate $l_{\vec{t}}(\vec{s})$ by

$$\sum_{\nu\in\mathbb{Z}} \int_{\mathbb{R}^n} \left(\sum_{Q\in Q_\nu} \left(|Q|^{-\frac{\alpha}{n}-\frac{1}{2}} \chi_Q(x) \, \|W^{1/p}(x)\vec{s}_Q\|_{\mathcal{H}} \right)^p \right)^{\frac{1}{p}}$$

$$\times \left(\sum_{Q\in Q_\nu} \left(|Q|^{\frac{\alpha}{n}-\frac{1}{2}} \chi_Q(x) \, \|W^{-1/p}(x)\vec{t}_Q\|_{\mathcal{H}} \right)^{p'} \right)^{\frac{1}{p'}} dx$$

$$\le \sum_{\nu\in\mathbb{Z}} \left\| \sum_{Q\in Q_\nu} |Q|^{-\frac{\alpha}{n}-\frac{1}{2}} \chi_Q \vec{s}_Q \right\|_{L^p(W)} \left\| \sum_{Q\in Q_\nu} |Q|^{\frac{\alpha}{n}-\frac{1}{2}} \chi_Q \vec{t}_Q \right\|_{L^{p'}(W^{-p'/p})} \tag{13}$$

$$\le \|\vec{s}\|_{\dot{b}_p^{\alpha q}(W)} \, \|\vec{t}\|_{\dot{b}_{p'}^{-\alpha q'}(W^{-p'/p})},$$

for $1 < q < \infty$. In case of $0 < q \le 1$, we bound (13) by $\|\vec{s}\|_{\dot{b}_p^{\alpha 1}(W)} \, \|\vec{t}\|_{\dot{b}_{p'}^{-\alpha\infty}(W^{-p'/p})}$. Since l^q is embedded into l^1 when $0 < q \le 1$, we estimate the previous product by $\|\vec{s}\|_{\dot{b}_p^{\alpha q}(W)} \, \|\vec{t}\|_{\dot{b}_{p'}^{-\alpha\infty}(W^{-p'/p})}$. □

In terms of reducing operators (or using (4)) the previous lemma states

$$\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{\#}\}) \subseteq \left[\dot{b}_p^{\alpha q}(\{A_Q\})\right]^*. \tag{14}$$

If we follow along the lines of the proof again but instead of $W^{-1/p}(t)\, W^{1/p}(t)$ in (12) use $A_Q^{-1}\, A_Q$, then we obtain the following statement.

Lemma 3.2. *Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \le p < \infty$ and $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$. Then*

$$\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \subseteq \left[\dot{b}_p^{\alpha q}(\{A_Q\})\right]^*. \tag{15}$$

In fact, if we have only proven (15), then (14) (and equivalently part (i) of Theorem 3.1) could have been obtained as a consequence of (15) and [5, Corollary 7.4], i.e.,

$$\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{\#}\}) \subseteq \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \subseteq \left[\dot{b}_p^{\alpha q}(\{A_Q\})\right]^*. \tag{16}$$

Observe that (15) holds for any $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$, not necessarily generated by W.

Now we will study the opposite embeddings. By Lemma 3.3 below, we will get

$$\left[\dot{b}_p^{\alpha q}(\{A_Q\})\right]^* \subseteq \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \tag{17}$$

without any additional assumptions on the sequence $\{A_Q\}_Q$. Note that combining (15) and (17), we obtain (7). Applying [5, Corollary 7.4] again, (17) is continued as

$$\left[\dot{b}_p^{\alpha q}(\{A_Q\})\right]^* \subseteq \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \overset{A_p}{\subseteq} \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{\#}\}) \approx \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p})$$

with the second embedding being held under the A_p condition. Thus, the embedding (ii) of Theorem 3.1 holds if $W \in A_p$.

Lemma 3.3. *Let $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$, $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \le p < \infty$. Then (17) holds.*

Proof. Let $l \in \left[\dot{b}_p^{\alpha q}(\{A_Q\})\right]^*$. We show that there exists $\vec{t} \in \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ such that for any $\vec{s} \in \dot{b}_p^{\alpha q}(\{A_Q\})$

$$l(\vec{s}) = (\vec{s}, \vec{t}\,) = \sum_Q \left\langle \vec{s}_Q, \vec{t}_Q \right\rangle_{\mathcal{H}} \quad \text{and} \quad \|\vec{t}\,\|_{\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} \leq \|l\|.$$

Let $\vec{e}_J^{\,(k)}$ denote a vector-valued sequence enumerated by dyadic cubes such that in the k^{th} component (k^{th} row) of this vector the J^{th} entry (corresponding to the dyadic cube J) is equal to 1 and all other entries are zero:

$$\vec{e}_J^{\,(k)} = (..., \{0\}_Q, ..., \{...0...1_{J^{th} entry}...0...\}_Q - k^{th} row, ..., \{0\}_Q, ...)^T.$$

Now if $\vec{s}$ has only finitely many non-zero entries, i.e., $\vec{s} = \sum_{\{Q\}\, finite} \sum_{k=1}^m s_Q^{(k)}\, \vec{e}_Q^{\,(k)}$, then by linearity

$$l(\vec{s}) = \sum_{\{Q\}\, finite} \sum_{k=1}^m s_Q^{(k)}\, l(\vec{e}_Q^{\,(k)}) =: \sum_{\{Q\}\, finite} \sum_{k=1}^m s_Q^{(k)}\, t_Q^{(k)}.$$

By continuity, since finitely non-zero sequences are dense ($p, q < \infty$), we get

$$l(\vec{s}) = \sum_{Q \in \mathcal{D}} \sum_{k=1}^m s_Q^{(k)} t_Q^{(k)} = \sum_{Q \in \mathcal{D}} \left\langle \vec{s}_Q, \vec{t}_Q \right\rangle_{\mathcal{H}} \text{ for any } \vec{s} \in \dot{b}_p^{\alpha q}(\{A_Q\}).$$

Now everything is set up to show that $\vec{t} := \left(\{t_Q^{(1)}\}_Q, \{t_Q^{(2)}\}_Q, ..., \{t_Q^{(m)}\}_Q\right)^T \in \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$. For $\vec{\tilde{s}} \in \dot{b}_p^{\alpha q}(\mathbb{R}^m)$, set $\vec{s}_Q = A_Q^{-1}\vec{\tilde{s}}$ and define

$$\tilde{l}(\vec{\tilde{s}}\,) := l(\{A_Q^{-1}\vec{\tilde{s}}_Q\}_Q) = l(\{\vec{s}_Q\}_Q) = \sum_Q \left\langle \vec{s}_Q, \vec{t}_Q \right\rangle_{\mathcal{H}}$$
$$= \sum_Q \left\langle A_Q \vec{s}_Q, A_Q^{-1}\vec{t}_Q \right\rangle_{\mathcal{H}} = \sum_Q \left\langle \vec{\tilde{s}}_Q, \vec{\tilde{t}}_Q \right\rangle_{\mathcal{H}},$$

where $\vec{\tilde{t}}_Q = A_Q^{-1}\vec{t}_Q$. By above,

$$|\,\tilde{l}(\vec{\tilde{s}}\,)| \leq c\, \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} = c\, \|\{\vec{\tilde{s}}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\mathbb{R}^m)},$$

i.e., l induces a continuous linear functional $\tilde{l}$ on $\dot{b}_p^{\alpha q}(\mathbb{R}^m)$. By Lemma 3.4 below $\{\vec{\tilde{t}}_Q\}_Q \in \dot{b}_{p'}^{-\alpha q'}(\mathbb{R}^m)$. Since the inside $L^{p'}$-norm of the $\dot{b}_{p'}^{-\alpha q'}(\mathbb{R}^m)$-norm of $\vec{\tilde{t}}$ is

$$\left\| \sum_{Q \in Q_\nu} |Q|^{-\frac{1}{2}} \vec{\tilde{t}}_Q \chi_Q \right\|_{L^{p'}} = \left\| \sum_{Q \in Q_\nu} |Q|^{-\frac{1}{2}} \|A_Q^{-1}\vec{t}_Q\|_{\mathcal{H}} \chi_Q \right\|_{L^{p'}} = \left\| \sum_{Q \in Q_\nu} |Q|^{-\frac{1}{2}} \vec{t}_Q \chi_Q \right\|_{L^{p'}(\{A_Q^{-1}\})},$$

$\vec{t} \in \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ and the lemma is proved. ☐

Lemma 3.4. *Let* $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \leq p < \infty$. *Then*

$$\left[\dot{b}_p^{\alpha q}(\mathbb{R}^m)\right]^* \approx \dot{b}_{p'}^{-\alpha q'}(\mathbb{R}^m). \tag{18}$$

Proof. It suffices to show only the scalar case ($m = 1$) of (18), since $\vec{s} \in \dot{b}_p^{\alpha q}(\mathbb{R}^m)$ means that each component $s^{(i)}$ belongs to $\dot{b}_p^{\alpha q}$ and by making zero all but one of the components of an arbitrary $\vec{s}$ we obtain (18).

The embedding $\left[\dot{b}_p^{\alpha q}\right]^* \supseteq \dot{b}_{p'}^{-\alpha q'}$ is a trivial application of Hölder's inequality plus the embedding $\dot{b}_p^{\alpha q} \to \dot{b}_p^{\alpha 1}$ for $q < 1$, so we concentrate only on the opposite embedding.

Suppose $l \in \left[\dot{b}_p^{\alpha q}\right]^*$. Using linearity and continuity, l can be represented by some sequence $\{t_Q\}_Q$ as $l(s) = \sum_Q s_Q \bar{t}_Q$ for any $s = \{s_Q\} \in \dot{b}_p^{\alpha q}$ and

$$|l(s)| = \left|\sum_Q s_Q \bar{t}_Q\right| \leq \|l\| \, \|s\|_{\dot{b}_p^{\alpha q}}. \tag{19}$$

Case $q \geq 1$: For each $\nu \in \mathbb{Z}$ let $f_\nu(s)(x) = \sum_{Q \in Q_\nu} |Q|^{-\frac{\alpha}{n}-\frac{1}{2}} s_Q \chi_Q(x)$. Define a map $I : \dot{b}_p^{\alpha q} \longrightarrow l^q(L^p)$ by $I(s) = \{f_\nu(s)\}_{\nu \in \mathbb{Z}}$. Observe that $\|I(s)\|_{l^q(L^p)} = \|s\|_{\dot{b}_p^{\alpha q}}$, in other words, by the natural construction I is a linear isometry onto the subspace $I(\dot{b}_p^{\alpha q})$ of $l^q(L^p)$. Then l induces a continuous linear functional $\tilde{l}$ on $I(\dot{b}_p^{\alpha q}) \subseteq l^q(L^p)$ (continuous in $l^q(L^p)$-norm) by $\tilde{l}(I(s)) = l(s)$. Since $l^q(L^p)$ is a Banach space, by the Hahn-Banach Theorem $\tilde{l}$ extends to a continuous linear functional $\tilde{l}_{ext}$ on all of $l^q(L^p)$ with $\|\tilde{l}_{ext}\| = \|\tilde{l}\| \leq \|l\|$. Since $[l^q(L^p)]^* = l^{q'}(L^{p'})$, $\tilde{l}_{ext}$ is represented by a sequence $g = \{g_\nu\}_{\nu \in \mathbb{Z}} \in l^{q'}(L^{p'})$ with $\|g\| = \|\{g_\nu\}_\nu\|_{l^{q'}(L^{p'})} \leq \|l\|$ and

$$\sum_Q s_Q \bar{t}_Q = l(s) = \tilde{l}(\{f_\nu(s)\}) = \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}^n} f_\nu(s)(x) \bar{g}_\nu(x)\, dx, \qquad \text{for any } \vec{s} \in \dot{b}_p^{\alpha q},$$

or

$$\sum_Q s_Q \bar{t}_Q = \sum_{\nu \in \mathbb{Z}} \sum_{Q \in Q_\nu} |Q|^{-\frac{\alpha}{n}-\frac{1}{2}} s_Q \int_Q \bar{g}_\nu(x)\, dx.$$

Taking $s_Q = 0$ for all but one cube, we get $t_Q = |Q|^{-\frac{\alpha}{n}+\frac{1}{2}} < g_\nu >_Q$. Using Hölder's inequality, we have

$$\|t\|_{\dot{b}_{p'}^{-\alpha q'}} = \left\| \left\{ \left\| \sum_{Q \in Q_\nu} < g_\nu >_Q \chi_Q \right\|_{L^{p'}} \right\}_\nu \right\|_{l^{q'}} \leq \|\{\vec{g}_\nu\}_\nu\|_{l^{q'}(L^{p'})} \leq \|l\|.$$

Case $0 < q < 1$: Suppose $1 < p < \infty$. Fix $\nu \in \mathbb{Z}$ and let F_ν denote a finite collection of cubes from Q_ν. Set $\tau_\nu = \sum_{Q \in F_\nu} \left(|Q|^{\frac{\alpha}{n}-\frac{1}{2}+\frac{1}{p'}} |t_Q|\right)^{p'}$. Since the sum is finite, $\tau_\nu < \infty$. Let $s_Q = |Q|^{(\frac{\alpha}{n}-\frac{1}{2}+\frac{1}{p'})p'} |t_Q|^{p'-2} t_Q$, if $Q \in F_\nu$ and $t_Q \neq 0$; otherwise let $s_Q = 0$. Note that $\|\{s_Q\}_Q\|_{\dot{b}_p^{\alpha q}} = \tau_\nu^{1/p}$. Observe that $\sum_Q s_Q \bar{t}_Q = \tau_\nu$ and by (19) $\tau_\nu \leq \|l\| \, \|s\|_{\dot{b}_p^{\alpha q}} = \|l\| \, \tau_\nu^{1/p}$. Since τ_ν is finite, we get $\tau_\nu^{1/p'} \leq \|l\|$ and the estimate holds independently of the collection F_ν taken. Hence, we can pass to the limit from F_ν to

Q_ν. Then,

$$\|t\|_{\dot b_{p'}^{-\alpha\infty}} = \sup_{\nu\in\mathbb{Z}} \left(\sum_{Q\in Q_\nu} \left(|Q|^{\frac{\alpha}{n}-\frac{1}{2}+\frac{1}{p'}} |t_Q| \right)^{p'} \right)^{1/p'} = \sup_{\nu\in\mathbb{Z}} \tau_\nu^{1/p'} \le \|l\| \quad \text{or} \quad t \in \dot b_{p'}^{-\alpha\infty}.$$

Now assume $p=1$. Fix $P\in\mathcal{D}$ and set $s^{(P)} = \left\{ s_Q^{(P)} \right\}_Q$ by $s_Q^{(P)} = |Q|^{\frac{\alpha}{n}-\frac{1}{2}} \operatorname{sgn} \bar t_Q$ if $Q = P$ and $s_Q^{(P)} = 0$ otherwise. Then $\left\| \left\{ s_Q^{(P)} \right\}_Q \right\|_{\dot b_1^{\alpha q}} = 1$ and $|P|^{\frac{\alpha}{n}-\frac{1}{2}} |t_P| = \sum_Q s_Q^{(P)} \bar t_Q = l\left(s^{(P)}\right) \le \|l\| \, \|\{s_Q^{(P)}\}_Q\|_{\dot b_1^{\alpha q}} = \|l\|$ for any $P\in\mathcal{D}$. Hence,

$$\|t\|_{\dot b_\infty^{-\alpha\infty}} = \sup_{P\in\mathcal{D}} |P|^{\frac{\alpha}{n}-\frac{1}{2}} |t_P| \le \|l\| \quad \text{or} \quad t\in \dot b_\infty^{-\alpha\infty}.$$

□

4. Equivalence of sequence and discrete averaging Besov spaces

In this section we discuss norm equivalence between $\dot B_p^{\alpha q}(\{A_Q\})$ and $\dot b_p^{\alpha q}(\{A_Q\})$. We suppose $\alpha\in\mathbb{R}$, $0<q\le\infty$ and $1\le p<\infty$ for all statements in this section. If $q=\infty$, then set $q'=1$.

Definition 4.1. For $\nu\in\mathbb{Z}$ let $E_\nu = \{\vec f : f_i \in \mathcal{S}' \text{ with } \operatorname{supp} \hat f_i \subseteq \{\xi\in\mathbb{R}^n : |\xi| \le 2^{\nu+1}\}, i = 1, ..., m\}$.

The following decomposition of an exponential type function is a useful tool in studying the norm equivalence (for the proof the reader is referred to [3, p.55]):

Lemma 4.2. *Suppose* $g\in\mathcal{S}'(\mathbb{R}^n), h\in\mathcal{S}(\mathbb{R}^n)$ *and* $\operatorname{supp}\hat g, \operatorname{supp}\hat h \subseteq \{|\xi| < 2^\nu\pi\}$ *for some* $\nu\in\mathbb{Z}$. *Then*

$$(g*h)(x) = \sum_{k\in\mathbb{Z}^n} 2^{-\nu n} g(2^{-\nu}k) \; h(x - 2^{-\nu}k). \tag{20}$$

Let $\Gamma = \{\gamma\in\mathcal{S} : \hat\gamma = 1 \text{ on } \{\xi\in\mathbb{R}^n : |\xi|\le 2\} \text{ and } \operatorname{supp}\hat\gamma \subseteq \{\xi\in\mathbb{R}^n : |\xi| < \pi\}\}$. Define $\gamma_\nu(x) = 2^{\nu n}\gamma(2^\nu x)$ for $\nu\in\mathbb{Z}$. Since $\hat\gamma_\nu = \hat\gamma(2^\nu\xi)$, $\operatorname{supp}\hat\gamma_\nu \subseteq \{\xi\in\mathbb{R}^n : |\xi| < 2^\nu\pi\}$.

Lemma 4.3. *For* $\nu\in\mathbb{Z}$ *let* $\vec g\in E_\nu$ *and fix* $x\in Q_{\nu k}$ *where* $k\in\mathbb{Z}^n$. *Then for any* $y\in\mathbb{R}^n$ *and* $\gamma\in\Gamma$

$$\vec g(y) = \sum_{l\in\mathbb{Z}^n} 2^{-\nu n} \vec g(2^{-\nu}l + x) \; \gamma_\nu(y - (2^{-\nu}l + x)). \tag{21}$$

Proof. Denote $\vec g^x(y) = \vec g(y+x)$. Trivially, $\vec g(y) = \vec g^x(y-x)$. Note that $(\vec g^x)\hat{}(\xi) = e^{ix\xi}\hat{\vec g}(\xi)$, and so $\operatorname{supp}(\vec g^x)\hat{} = \operatorname{supp}\hat{\vec g}$. Therefore, by (20) applied to $\vec g^x$:

$$\vec g(y) = \vec g^x(y-x) = \sum_{l\in\mathbb{Z}^n} 2^{-\nu n} \vec g^x(2^{-\nu}l) \; \gamma_\nu(y - x - 2^{-\nu}l),$$

which is (21). □

Lemma 4.4. *If* $\{A_Q\}$ *is a doubling sequence of order p, then for* $\vec s_Q = \left\langle \vec f, \varphi_Q \right\rangle$

$$\|\{\vec s_Q\}_Q\|_{\dot b_p^{\alpha q}(\{A_Q\})} \le c\, \|\vec f\|_{\dot B_p^{\alpha q}(\{A_Q\})}. \tag{22}$$

Proof. Note that $\vec{s}_Q = |Q|^{1/2}(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}k)$ for $Q = Q_{\nu k}$, where $\tilde{\varphi}(x) = \overline{\varphi(-x)}$. Let $\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} =: \big\|\{J_\nu^{1/p}\}_\nu\big\|_{l_q^\alpha}$, where

$$J_\nu = \sum_{k\in\mathbb{Z}^n} \int_{Q_{\nu k}} \|A_{Q_{\nu k}}(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}k)\|^p \, dx. \tag{23}$$

Since $\tilde{\varphi}_\nu * \vec{f} \in E_\nu$, Lemma 4.3 implies

$$(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}k) = \sum_{l\in\mathbb{Z}^n} (\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}l + x)\, \gamma(k - l - 2^\nu x), \ x \in Q_{\nu k}$$

for some $\gamma \in \Gamma$. Then

$$\begin{aligned} J_\nu &\le \sum_{k\in\mathbb{Z}^n} \int_{Q_{\nu k}} \left(\sum_{l\in\mathbb{Z}^n} \|A_{Q_{\nu k}}(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}l + x)\| \, |\gamma(k-l-2^\nu x)| \right)^p dx \\ &\le c \sum_{k\in\mathbb{Z}^n} \int_{Q_{\nu k}} \left(\sum_{l\in\mathbb{Z}^n} \frac{\|A_{Q_{\nu k}}(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}l + x)\|}{(1 + |k-l-2^\nu x|^M} \right)^p dx \quad \text{for some } M > \beta + n. \end{aligned}$$

Using the discrete Hölder inequality and the fact that $M > n$, we bring the p^{th} power inside the sum on l (for $p > 1$). Furthermore, since $\{A_Q\}_Q$ is doubling, (1) implies

$$\|A_{Q_{\nu k}}\vec{u}\,\|^p \le c\,(1+|l|)^\beta \, \|A_{Q_{\nu(k+l)}}\vec{u}\,\|^p, \quad \text{for any } \vec{u} \in \mathcal{H}. \tag{24}$$

Thus,

$$J_\nu \le c \sum_{k\in\mathbb{Z}^n} \int_{Q_{\nu k}} \sum_{l\in\mathbb{Z}^n} \frac{(1+|l|)^\beta \|A_{Q_{\nu(k+l)}}(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}l + x)\|^p}{(1+|k-l-2^\nu x|)^M} \, dx.$$

Changing variable ($t = x + 2^{-\nu}l$) and reindexing the sum on l, we get

$$\begin{aligned} J_\nu &\le c \sum_{k\in\mathbb{Z}^n} \int_{Q_{\nu l}} \sum_{l\in\mathbb{Z}^n} (1+|k-l|)^{\beta - M} \|A_{Q_{\nu l}}(\tilde{\varphi}_\nu * \vec{f})(t)\|^p \, dt \\ &\le c \sum_{l\in\mathbb{Z}^n} \int_{Q_{\nu l}} \|A_{Q_{\nu l}}(\tilde{\varphi}_\nu * \vec{f})(t)\|^p \, dt = c\, \|\tilde{\varphi}_\nu * \vec{f}\,\|_{L^p(\{A_Q\},\nu)} \end{aligned}$$

(the sum on k converges since $M - \beta > n$). Thus,

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} \le c\, \|\vec{f}\,\|_{\dot{B}_p^{\alpha q}(\{A_Q\},\tilde{\varphi})}.$$

Now we need the independence of the space $\dot{B}_p^{\alpha q}(\{A_Q\})$ on the choice of φ (or $\tilde{\varphi}$). We apply the same strategy as in [5, Theorem 6.6], namely, we use the proof of Corollary 4.9 below, which will imply that the last expression is equivalent to $c\,\|\vec{f}\,\|_{\dot{B}_p^{\alpha q}(\{A_Q\},\varphi)}$ and, thus, (22) is proved. ☐

Corollary 4.5. *If $\{A_Q\}$ is a doubling sequence of order p, then for $\vec{s}_Q = \left\langle \vec{f}, \varphi_Q \right\rangle$*

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q^{-1}\})} \le c\, \|\vec{f}\,\|_{\dot{B}_p^{\alpha q}(\{A_Q^{-1}\})} \tag{25}$$

and

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} \le c\, \|\vec{f}\,\|_{\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})}. \tag{26}$$

Proof. For (25) repeat the previous proof with each A_Q replaced by A_Q^{-1} and instead of the estimate (24) use

$$\|A_{Q_{\nu k}}^{-1}\vec{u}\,\|^p \le c\,(1+|l|)^\beta\,\|A_{Q_{\nu(k+l)}}^{-1}\vec{u}\,\|^p, \quad \text{for any } \vec{u}\in\mathcal{H}, \tag{27}$$

which follows from the doubling property (1) and duality $\|A_Q^{-1}\vec{u}\,\| = \sup\limits_{\vec{v}\neq 0}\dfrac{|(\vec{u},\vec{v}\,)|}{\|A_Q\vec{v}\,\|}$.

For (26) use the obvious replacements for α, p, q and A_Q. If $1<p<\infty$, choose $M>\beta p'/p+n$ and replace (24) by

$$\|A_{Q_{\nu k}}^{-1}\vec{u}\,\|^{p'} \le c\,(1+|l|)^{\beta p'/p}\,\|A_{Q_{\nu(k+l)}}^{-1}\vec{u}\,\|^{p'}, \quad \text{for any } \vec{u}\in\mathcal{H}, \tag{28}$$

which is obtained from (27) by raising to the power p'/p. If $p=1$ ($p'=\infty$), then replace (23) with the L^∞-norm:

$$J_\nu = \sup_{x\in\mathbb{R}^n}\sum_{k\in\mathbb{Z}^n}\|A_{Q_{\nu k}}^{-1}(\tilde{\varphi}_\nu * \vec{f}\,)(2^{-\nu}k)\|\,\chi_{Q_{\nu k}}(x)$$

and use (27) instead of (24) to get

$$J_\nu \le c\sup_{t\in\mathbb{R}^n}\sum_{l\in\mathbb{Z}^n}\|A_{Q_{\nu l}}^{-1}(\tilde{\varphi}_\nu * \vec{f}\,)(t)\|\,\chi_{Q_{\nu l}}(t) = c\,\|\tilde{\varphi}_\nu * \vec{f}\,\|_{L^\infty(\{A_Q^{-1}\},\nu)}.$$

□

Recall that for each admissible $\varphi\in\mathcal{A}$ there exists $\psi\in\mathcal{A}$ (see [3, Lemma 6.9]) such that

$$\sum_{\nu\in\mathbb{Z}}\overline{\hat{\varphi}(2^\nu\xi)}\cdot\hat{\psi}(2^\nu\xi)=1, \text{ if } \xi\neq 0. \tag{29}$$

A pair (φ,ψ) with $\varphi,\psi\in\mathcal{A}$ and the property (29) is referred to as a pair of mutually admissible kernels.

Lemma 4.6. *Suppose $\{A_Q\}_Q$ is a doubling sequence of order p. Then*

$$\|\vec{f}\,\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} \le c\left\|\left\{\vec{s}_Q(\vec{f}\,)\right\}_Q\right\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}, \tag{30}$$

Proof. Using $\vec{f} = \sum\limits_Q \vec{s}_Q(\vec{f}\,)\psi_Q$, we get

$$\left\|\sum_Q \vec{s}_Q(\vec{f}\,)\psi_Q\right\|_{\dot{B}_p^{\alpha q}(\{A_Q\})}$$

$$\le\left\|\left\{\sum_{\mu\in\mathbb{Z}}\left(\sum_{l(P)=2^{-\nu}}\int_P\left(\sum_{l(Q)=2^{-\mu}}\|A_P\vec{s}_Q\||(\varphi_\nu*\psi_Q)(x)|\right)^p dx\right)^{1/p}\right\}_\nu\right\|_{l_q^\alpha}$$

$$=\left\|\left\{\sum_{\mu=\nu-1}^{\nu+1}\left(\sum_{l(P)=2^{-\nu}}\int_P\left(\sum_{l(Q)=2^{-\mu}}\|A_P\vec{s}_Q\||(\varphi_\nu*\psi_Q)(x)|\right)^p dx\right)^{1/p}\right\}_\nu\right\|_{l_q^\alpha}$$

$$=: \left\| \left\{ J_\nu^{1/p} \right\}_\nu \right\|_{l_q^\alpha},$$

since $\varphi_\nu * \psi_Q = 0$ if $|\mu - \nu| > 1$. Using the convolution estimates (16) and (17) from [5], we get (for any $M > 0$)

$$|(\varphi_\nu * \psi_Q)(x)| \le c_M\, |Q|^{-1/2} (1 + 2^\nu |x - x_Q|)^{-M} \quad \text{if } \mu = \nu - 1, \nu, \nu + 1. \tag{31}$$

If $1 < p < \infty$, choose $M = M_1 + M_2$ with $M_1 > \beta/p + n/p$ and $M_2 > n/p'$; if $p = 1$, let $M = M_1 > \beta + n$. Then applying the above estimate and Hölder's inequality, we obtain

$$J_\nu \le c \sum_{\mu=\nu-1}^{\nu+1} \sum_{l(P)=2^{-\nu}} \sum_{l(Q)=2^{-\mu}} \|A_P \vec{s}_Q\|^p\, |P|\, |Q|^{-p/2} \left(1 + 2^\nu |x_P - x_Q|\right)^{-M_1 p}.$$

Shifting A_P to A_Q by doubling, we get

$$J_\nu \le c \sum_{\mu=\nu-1}^{\nu+1} \sum_{l(Q)=2^{-\mu}} |Q|^{-p/2}\, \|A_Q \vec{s}_Q\|^p\, |Q| \sum_{l(P)=2^{-\nu}} c_\beta \left(1 + 2^\nu |x_P - x_Q|\right)^{-M_1 p + \beta}.$$

Applying [5, Lemma 5.4] (Summation Lemma) to the sum on P, we have

$$J_\nu \le c \sum_{\mu=\nu-1}^{\nu+1} \sum_{l(Q)=2^{-\mu}} |Q|^{-p/2}\, \|A_Q \vec{s}_Q\|^p\, |Q| = c \sum_{\mu=\nu-1}^{\nu+1} \left\| \sum_{l(Q)=2^{-\mu}} |Q|^{-1/2}\, \vec{s}_Q \chi_Q \right\|_{L^p(\{A_Q\},\mu)}.$$

Combining the estimates for all J_ν and reindexing when necessary, we get

$$\left\| \vec{f} \right\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} \le 3\, c \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-1/2} \vec{s}_Q \chi_Q \right\|_{L^p(\{A_Q\},\nu)} \right\}_\nu \right\|_{l^q} = c\, \|\{\vec{s}_Q\}\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}.$$

□

Remark 4.7. Theorem 1.6 is obtained by combining Lemmas 4.4 and 4.6.

Corollary 4.8. *If $\{A_Q\}_Q$ is doubling (of order p), then*

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q^{-1}\})} \le c \left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(\{A_Q^{-1}\})} \tag{32}$$

and

$$\|\vec{f}\|_{\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} \le c \left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_Q \right\|_{\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})}. \tag{33}$$

Proof. For (32) use the previous proof with the following shifting of A_P to A_Q (similar to (27)):

$$\|A_P^{-1} \vec{s}_Q\|^p \le c_{n,\beta,p} \left(1 + 2^\nu |x_P - x_Q|\right)^\beta \|A_Q^{-1} \vec{s}_Q\|^p, \tag{34}$$

where $l(P) = 2^{-\nu}$ and $l(Q) = 2^{-\mu}$ with $\mu = \nu - 1, \nu$ or $\nu + 1$; for (33) use the above proof with the indices $-\alpha$, q', p'; if $1 < p < \infty$, take $M > \beta p'/p + n$ and apply (34) raised to the power p'/p; if $p' = \infty$, then

$$J_\nu \le \sup_{x\in\mathbb{R}^n} \sum_{\mu=\nu-1}^{\nu+1} \sum_{l(P)=2^{-\nu}} \sum_{l(Q)=2^{-\mu}} \|A_P^{-1}\vec{s}_Q\| \, |(\varphi_\nu * \psi_Q)(x)| \, \chi_P(x).$$

Using the convolution estimate (31) (with $M = M_1 > \beta + n$) and (34) for shifting A_P^{-1} to A_Q^{-1}, we get

$$J_\nu \le c \sum_{\mu=\nu-1}^{\nu+1} \left\| \sum_{l(Q)=2^{-\mu}} \left\|A_Q^{-1}\vec{s}_Q\right\| \, \chi_Q \right\|_{L^\infty},$$

which gives (33). ☐

Corollary 4.9. *The spaces $\dot{B}_p^{\alpha q}(\{A_Q\})$, $\dot{B}_p^{\alpha q}(\{A_Q^{-1}\})$ and $\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ are independent of the choice of the admissible kernel, if $\{A_Q\}_Q$ is doubling (of order p).*

Proof. Repeat the proof of [5, Theorem 1.8] with W replaced by A_Q and use Lemmas 4.4 and 4.6 for the space $\dot{B}_p^{\alpha q}(\{A_Q\})$; for the space $\dot{B}_p^{\alpha q}(\{A_Q^{-1}\})$ apply (25) and (32), and for the space $\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ use (26) and (33). ☐

5. Properties of averaging L^p spaces

In this section we study the connection between $L^p(\{A_Q\},\nu)$ and $L^p(W)$, the dual of $L^p(\{A_Q\},\nu)$ and several convolution estimates on $L^p(\{A_Q\},\nu)$.

Lemma 5.1. *Let W be a doubling matrix weight of order p, $1 \le p < \infty$. Then for $\vec{f} \in E_\nu$, $\nu \in \mathbb{Z}$*

$$\|\vec{f}\|_{L^p(W)} \le c \, \|\vec{f}\|_{L^p(\{A_Q\},\nu)}, \tag{35}$$

where $\{A_Q\}_Q$ is a sequence of reducing operators generated by W and c is independent of ν.

Proof. Using the notation $W_\nu(t) = W(2^{-\nu}t)$ and $\vec{f}_\nu(t) = \vec{f}(2^{-\nu}t)$, we write

$$\|\vec{f}\|^p_{L^p(W)} = \sum_{k\in\mathbb{Z}^n} \int_{Q_{\nu k}} \|W^{1/p}(t)\vec{f}(t)\|^p \, dt = \sum_{k\in\mathbb{Z}^n} 2^{-\nu n} \int_{Q_{0k}} \|W_\nu^{1/p}(t)\vec{f}_\nu(t)\|^p \, dt.$$

Since $\vec{f}_\nu \in E_0$, there exists $\gamma \in \Gamma$ such that $\vec{f}_\nu = \vec{f}_\nu * \gamma$. Using the decay of γ and Hölder's inequality, we get

$$\|\vec{f}\|^p_{L^p(W)} \le \sum_{k\in\mathbb{Z}^n} 2^{-\nu n} \int_{Q_{0k}} \sum_{m\in\mathbb{Z}^n} \int_{Q_{0m}} \frac{\|W_\nu^{1/p}(t)\vec{f}_\nu(y)\|^p}{(1+|m-k|)^M} \, dy \, dt,$$

for some $M > \beta + n$. Observe that $\|A_{Q_{\nu k}}\vec{f}_\nu(y)\|^p \approx \int_{Q_{0k}} \|W_\nu^{1/p}(t)\vec{f}_\nu(y)\|^p \, dt$. Using the doubling property of W to shift $A_{Q_{\nu k}}$ to $A_{Q_{\nu m}}$ (see (24)), we obtain

$$\|\vec{f}\|^p_{L^p(W)} \le c \sum_{m\in\mathbb{Z}^n} \sum_{k\in\mathbb{Z}^n} 2^{-\nu n} \int_{Q_{0m}} (1+|m-k|)^{-(M-\beta)} \|A_{Q_{\nu m}}\vec{f}_\nu(y)\|^p \, dy$$

$$\leq c \sum_{m\in\mathbb{Z}^n} \int_{Q_{0m}} \|A_{Q_{\nu m}} \vec{f}_\nu(y)\|^p \, dy,$$

where the sum on k converges, since $M > \beta + n$. Changing variables $x = 2^{-\nu}y$, we get the desired inequality (35). □

Corollary 5.2. *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$ and $1 \leq p < \infty$. If W is doubling (of order p) and $\{A_Q\}_Q$ is a sequence of reducing operators generated by W, then*

$$\dot{B}_p^{\alpha q}(\{A_Q\}) \subseteq \dot{B}_p^{\alpha q}(W).$$

Proof. Since $\varphi_\nu * \vec{f} \in E_\nu$, the previous lemma implies

$$\left\|\vec{f}\right\|_{\dot{B}_p^{\alpha q}(W)} = \left\|\left\{2^{\nu\alpha}\left\|\varphi_\nu * \vec{f}\right\|_{L^p(W)}\right\}_\nu\right\|_{l^q}$$

$$\leq c \left\|\left\{2^{\nu\alpha}\left\|\varphi_\nu * \vec{f}\right\|_{L^p(\{A_Q\},\nu)}\right\}_\nu\right\|_{l^q} = c\left\|\vec{f}\right\|_{\dot{B}_p^{\alpha q}(\{A_Q\})}.$$

□

Lemma 5.3. *Let $1 \leq p < \infty$ and W satisfies any of (A1)-(A3). Suppose $\vec{f} \in E_\nu$, $\nu \in \mathbb{Z}$. Then*

$$\left\|\vec{f}\right\|_{L^p(\{A_Q\},\nu)} \leq c\left\|\vec{f}\right\|_{L^p(W)}, \tag{36}$$

where $\{A_Q\}_Q$ is a sequence of reducing operators produced by W and c is independent of ν.

Proof. Using the definition of reducing operators, we write

$$\|\vec{f}\|_{L^p(\{A_Q\},\nu)} \approx \sum_{k\in\mathbb{Z}^n} \int_{Q_{\nu k}} \frac{1}{|Q_{\nu k}|} \int_{Q_{\nu k}} \|W^{1/p}(t)\vec{f}(x)\,\|^p \, dt\, dx$$

$$= \sum_{k\in\mathbb{Z}^n} \int_{Q_{0k}} \int_{Q_{\nu k}} \|W^{1/p}(t)\vec{f}_\nu(y)\,\|^p \, dt\, dy,$$

by changing variables $x = 2^{-\nu}y$ and denoting $\vec{f}_\nu(y) = \vec{f}(2^{-\nu}y)$. Note that $\vec{f}_\nu \in E_0$. Applying the decomposition of an exponential type function (Lemma 4.2) to $\vec{f}_\nu = \vec{f}_\nu * \gamma$ for $\gamma \in \Gamma$ and Hölder's inequality (choose $M > \beta + n$), the last expression is bounded by

$$c \sum_{k\in\mathbb{Z}^n} \int_{Q_{0k}} \int_{Q_{\nu k}} \sum_{m\in\mathbb{Z}^n} \frac{\|W^{1/p}(t)\vec{f}_\nu(m)\|^p}{(1+|y-m|)^M} \, dt\, dy$$

$$\leq c \sum_{m\in\mathbb{Z}^n} \sum_{k\in\mathbb{Z}^n} \frac{1}{(1+|k-m|)^{M-\beta}} \int_{Q_{0k}} \int_{Q_{\nu m}} \|W^{1/p}(t)\vec{f}_\nu(m)\|^p \, dt\, dy,$$

by applying the doubling property of W (any of (A1)-(A3) imply that W is doubling). Integrating on y and summing on k ($M > \beta + n$), we bound the previous line by

$$c \sum_{m\in\mathbb{Z}^n} \int_{Q_{\nu m}} \|W^{1/p}(t)\vec{f}_\nu(m)\|^p \, dt = c\, 2^{-\nu n} \sum_{m\in\mathbb{Z}^n} \int_{Q_{0m}} \|W_\nu^{1/p}(t)\vec{f}_\nu(m)\|^p \, dt,$$

again by changing variables. Now applying [5, Lemma 6.3] and [5, Lemma 6.5] (this is where (A1)-(A3) come into play), we bound the above by $c\,2^{-\nu n}\,\|\vec{f}_\nu\|^p_{L^p(W_\nu)} = c\,\|\vec{f}\|^p_{L^p(W)}$, which gives (36). ☐

Corollary 5.4. *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$ and $1 \leq p < \infty$. If W satisfies any of (A1)-(A3) and $\{A_Q\}_Q$ is a sequence of reducing operators generated by W, then*

$$\dot{B}^{\alpha q}_p(W) \subseteq \dot{B}^{\alpha q}_p(\{A_Q\}).$$

Proof. As in the proof of Corollary 5.2, use the fact that $\varphi_\nu * \vec{f} \in E_\nu$ and Lemma 5.3. ☐

Remark 5.5. Combining Corollaries 5.2 and 5.4, we have Lemma 1.7.

In order to establish the dual of $L^p(\{A_Q\},\nu)$, $1 < p < \infty$, we consider the following idea:

$$\begin{aligned}\left\|\vec{f}\right\|^p_{L^p(\{A_Q\},\nu)} &= \sum_{Q\in Q_\nu}\int_Q \|A_Q\vec{f}(x)\|^p_{\mathcal{H}}\,dx = \int_{\mathbb{R}^n}\left(\sum_{Q\in Q_\nu}\|A_Q\vec{f}(x)\|_{\mathcal{H}}\chi_Q(x)\right)^p dx\\ &= \int_{\mathbb{R}^n}\left\|\sum_{Q\in Q_\nu}A_Q\chi_Q(x)\vec{f}(x)\right\|^p_{\mathcal{H}} dx =: \int_{\mathbb{R}^n}\left\|U_\nu^{1/p}(x)\vec{f}(x)\right\|^p_{\mathcal{H}} dx = \left\|\vec{f}\right\|^p_{L^p(U_\nu)},\end{aligned}$$

i.e., $L^p(\{A_Q\},\nu) = L^p(U_\nu)$, where $U_\nu(x) = \sum\limits_{Q\in Q_\nu} A_Q^p\chi_Q(x)$ is a matrix weight. Since the dual $[L^p(U_\nu)]^*$ can be identified with $L^{p'}(U_\nu^*)$ with $U_\nu^{1/p}(x) = (U_\nu^*)^{-1/p'}(x)$ (e.g. see [4] or [10]), i.e., $U_\nu^*(x) = \sum\limits_{Q\in Q_\nu} A_Q^{-p'}\chi_Q(x)$, we obtain

$$\begin{aligned}\left\|\vec{f}\right\|^{p'}_{L^{p'}(U_\nu^*)} &= \int_{\mathbb{R}^n}\left\|\sum_{Q\in Q_\nu}A_Q^{-1}\chi_Q(x)\vec{f}(x)\right\|^{p'}_{\mathcal{H}} dx\\ &= \sum_{Q\in Q_\nu}\int_Q\left\|A_Q^{-1}\vec{f}(x)\right\|^{p'}_{\mathcal{H}} dx = \left\|\vec{f}\right\|^{p'}_{L^{p'}(\{A_Q^{-1}\},\nu)},\end{aligned}$$

or

$$[L^p(\{A_Q\},\nu)]^* \approx L^{p'}(\{A_Q^{-1}\},\nu). \tag{37}$$

If $p = 1$, then the standard duality argument gives $[L^1(\{A_Q\},\nu)]^* \approx L^\infty(\{A_Q^{-1}\},\nu)$. The details are left to the reader.

The boundedness of the convolution operator with a decaying kernel on $L^p(\{A_Q\},\nu)$ will be helpful in the next section. We establish it here.

Lemma 5.6. *Let $|\Phi(t)| \leq \dfrac{c}{(1+|t|)^M}$ for some $M > \beta/p + n$ and for $\nu \in \mathbb{Z}$ define $\Phi_\nu(t) = 2^{\nu n}\Phi(2^\nu t)$. Let $\{A_Q\}_Q$ be a doubling matrix sequence of order p, $1 \leq p < \infty$. Fix $\lambda, \mu, \nu \in \mathbb{Z}$. Then*

(i) $\|\Phi_\mu * \vec{f}\|_{L^p(\{A_Q\},\lambda)} \leq c_0\,(c_1)^{\lambda-\nu}(c_2)^{\mu-\nu}\|\vec{f}\|_{L^p(\{A_Q\},\nu)}$,

(ii) $\|\Phi_\mu * \vec{f}\|_{L^p(\{A_Q^{-1}\},\lambda)} \leq c_0\,(c_3)^{\lambda-\nu}(c_2)^{\mu-\nu}\|\vec{f}\|_{L^p(\{A_Q^{-1}\},\nu)}$, *where* $c_1 = 2^{n/p}\chi_{\{\lambda>\nu\}} + 2^{(n-\beta)/p}\chi_{\{\lambda\leq\nu\}}$, $c_2 = 2^{n}\chi_{\{\mu>\nu\}} + 2^{n-M}\chi_{\{\mu\leq\nu\}}$, $c_3 = 2^{(\beta-n)/p}\chi_{\{\lambda>\nu\}} + 2^{-n/p}\chi_{\{\lambda\leq\nu\}}$, *and* c_0 *is independent of* λ, μ *and* ν.

Proof. Using the decay of Φ, namely, $|\Phi_\mu(x-y)| \leq c\,k_2 \dfrac{2^{\nu n}}{(1+2^\nu|x-y|)^M}$, where $k_2 = 2^{(\mu-\nu)n}\chi_{\{\mu>\nu\}} + 2^{(\nu-\mu)(M-n)}\chi_{\{\mu\leq\nu\}}$, we have

$$\begin{aligned}
\|\Phi_\mu * \vec{f}\,\|^p_{L^p(\{A_Q\},\lambda)} &= \sum_{Q\in Q_\lambda}\int_Q \|A_Q(\Phi_\mu*\vec{f}\,)(x)\|^p\,dx \\
&\leq \sum_{Q\in Q_\lambda}\int_Q\left(\int_{\mathbb{R}^n}\|A_Q\vec{f}(y)\,\|\,|\Phi_\mu(x-y)|\,dy\right)^p dx \\
&\leq c\sum_{Q\in Q_\lambda}\int_Q\left(\int_{\mathbb{R}^n}\frac{k_2\,2^{\nu n}\|A_Q\vec{f}(y)\,\|}{(1+2^\nu|x-y|)^M}\,dy\right)^p dx \\
&\approx c\sum_{k\in\mathbb{Z}^n}\int_{Q_{\lambda k}}\left(\sum_{m\in\mathbb{Z}^n}\int_{Q_{\nu m}}\frac{k_2\,2^{\nu n}\|A_{Q_{\lambda k}}\vec{f}(y)\|}{(1+2^\nu|x-x_{Q_{\nu m}}|)^M}\,dy\right)^p dx.
\end{aligned}$$

Since $\{A_Q\}_Q$ is doubling, we "shift" $A_{Q_{\lambda k}}$ to $A_{Q_{\nu m}}$:

$$\|A_{Q_{\lambda k}}\vec{f}(y)\,\| \leq c\,k_1(1+2^\nu|x-x_{Q_{\nu m}}|)^{\beta/p}\,\|A_{Q_{\nu m}}\vec{f}(y)\,\| \quad \text{for } x\in Q_{\lambda k}, \tag{38}$$

where $k_1 = 2^{(\lambda-\nu)n/p}\chi_{\{\lambda>\nu\}} + 2^{(\nu-\lambda)(\beta-n)/p}\chi_{\{\lambda\leq\nu\}}$. Substituting (38) into the convolution estimate, we get

$$\|\Phi_\mu * \vec{f}\,\|^p_{L^p(\{A_Q\},\lambda)} \leq c\int_{\mathbb{R}^n}\left(\sum_{m\in\mathbb{Z}^n}\int_{Q_{\nu m}}\frac{k_1k_2 2^{\nu n}\|A_{Q_{\nu m}}\vec{f}(y)\|}{(1+|2^\nu x-m|)^{M-\beta/p}}\,dy\right)^p dx.$$

Using the discrete Hölder inequality on the sum inside and then Jensen's inequality to bring p^{th} power inside of the integral (if $p>1$), the last line is bounded above by

$$\begin{aligned}
&c\,k_1^p\,k_2^p\int_{\mathbb{R}^n}\left(\sum_{l\in\mathbb{Z}^n}\frac{1}{(1+|2^\nu x-l|)^{M-\beta/p}}\right)^{p/p'}\left(\sum_{m\in\mathbb{Z}^n}\int_{Q_{\nu m}}\frac{2^{\nu n}\|A_{Q_{\nu m}}\vec{f}(y)\|^p}{(1+|2^\nu x-m|)^{M-\beta/p}}\,dy\right)dx \\
&\qquad\leq c\,k_1^p\,k_2^p\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\frac{2^{\nu n}}{(1+|2^\nu x-m|)^{M-\beta/p}}\int_{Q_{\nu m}}\|A_{Q_{\nu m}}\vec{f}(y)\|^p\,dy\,dx,
\end{aligned}$$

since $M-\beta/p>n$, the sum on l converges (independently of x). Changing variables ($t=2^\nu x$) and observing that the integral on t converges (independently of m), again since $M-\beta/p>n$, we obtain

$$\|\Phi_\mu * \vec{f}\,\|^p_{L^p(\{A_Q\},\mu)} \leq c\,k_1^p\,k_2^p\sum_{m\in\mathbb{Z}^n}\int_{Q_{\nu m}}\|A_{Q_{\nu m}}\vec{f}(y)\|^p\,dy.$$

Put $c_1 = k_1^{1/(\lambda-\nu)}$ and $c_2 = k_2^{1/(\mu-\nu)}$. Then part (i) is proved.

For the second part observe that (1) ("shift" $A_{Q_{\nu m}}$ to $A_{Q_{\lambda k}}$) together with $\|A_Q^{-1}\vec{v}\,\| = \sup\limits_{\vec{u}\neq 0}\dfrac{|(\vec{v},\vec{u}\,)|}{\|A_Q\vec{u}\,\|}$ imply

$$\|A^{-1}_{Q_{\lambda k}}\vec{f}(y)\,\| \leq c\,k_3(1+2^\nu|x-x_{Q_{\nu m}}|)^{\beta/p}\|A^{-1}_{Q_{\nu m}}\vec{f}(y)\,\|, \quad x\in Q_{\lambda k}, \tag{39}$$

where $k_3 = 2^{(\lambda-\nu)(\beta-n)/p}\chi_{\{\lambda>\nu\}} + 2^{(\nu-\lambda)n/p}\chi_{\{\lambda\leq\nu\}}$. Note that (39) is similar to (38), so previous estimates with each A_Q replaced by A_Q^{-1} prove (ii) with $c_3 = k_3^{1/(\lambda-\nu)}$. $\square$

Remark 5.7. Recall that $\|A_Q^{-1}\vec{u}\,\| \leq c\,\|A_Q^{\#}\vec{u}\,\|$ for any $\vec{u} \in \mathcal{H}$ (since $\|(A_Q^{\#}A_Q)^{-1}\| \leq c$). Suppose that $W^{-p'/p}$ is a doubling matrix of order p', $1 < p' < \infty$, with the doubling exponent β^* (instead of the assumption that W is doubling of order p). Then

$$\|A_{Q_{\lambda k}}^{-1}\vec{f}(y)\,\| \leq c\,\|A_{Q_{\lambda k}}^{\#}\vec{f}(y)\,\| \leq c\,k_1^*(1+2^{\nu}|x-x_{Q_{\nu m}}|)^{\beta^*/p'}\|A_{Q_{\nu m}}^{\#}\vec{f}(y)\,\|,$$

(where $k_1^* = 2^{(\lambda-\nu)n/p'}\chi_{\{\lambda>\nu\}} + 2^{(\nu-\lambda)(\beta^*-n)/p'}\chi_{\{\lambda\leq\nu\}}$, i.e., k_1 with β replaced by β^* and p by p') holds instead of (38). Choosing $M > \beta^*/p' + n$ in the previous lemma, we get

(iii) $\|\Phi_\mu * \vec{f}\,\|_{L^p(\{A_Q^{-1}\},\lambda)} \leq c_0\,(c_1^*)^{\lambda-\nu}(c_2)^{\mu-\nu}\,\|\vec{f}\,\|_{L^p(\{A_Q^{\#}\},\nu)}$, $1 < p < \infty$.

Remark 5.8. A similar convolution estimate can be proved for $L^p(W)$ spaces, $1 < p < \infty$:

$$\|\Phi * \vec{f}\,\|_{L^p(W)} \leq c\,\|\vec{f}\,\|_{L^p(W)}. \tag{40}$$

Recall that if Φ were to be a Calderón-Zygmund singular kernel K, then $\|K * \vec{f}\,\|_{L^p(W)} \leq c\,\|\vec{f}\,\|_{L^p(W)}$ if $W \in A_p$ (see [4], [9], [10]). Conversely, if (40) holds for every $\Phi \in \mathcal{S}$, then $W \in A_p$ is necessary (see [6]).

6. Duality of continuous Besov spaces

Now we shift our attention to continuous Besov spaces and our task is to construct $\left[\dot{B}_p^{\alpha q}(\{A_Q\})\right]^*$ and eventually $\left[\dot{B}_p^{\alpha q}(W)\right]^*$.

Lemma 6.1. *Let $\{A_Q\}_Q$ be a doubling matrix sequence of order p, $1 \leq p < \infty$. Let $\alpha \in \mathbb{R}$ and $0 < q < \infty$. Then*

$$\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \subseteq \left[\dot{B}_p^{\alpha q}(\{A_Q\})\right]^*. \tag{41}$$

Proof. Take $\varphi, \psi \in \mathcal{A}$ with the mutual property (29). Let $\tilde{\psi}(x) = \overline{\psi(-x)}$. Note that $\hat{\tilde{\psi}}(\xi) = \overline{\hat{\psi}}(\xi)$. Let $\vec{f} \in \dot{B}_p^{\alpha q}(\{A_Q\})$ and $\vec{g} \in \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$. First consider $\mathcal{S}_0 = \{f \in \mathcal{S} : 0 \notin \text{supp } \hat{f}\}$ a dense subspace of $\dot{B}_p^{\alpha q}(\{A_Q\})$ (see [6]) and take $\vec{f}$ with $(\vec{f})_i \in \mathcal{S}_0$, $i = 1, ..., m$ (and $\vec{g}$ with $(\vec{g})_i \in \mathcal{S}'$). Then

$$\vec{g} = \sum_{\nu\in\mathbb{Z}} \vec{g} * (\varphi_\nu * \tilde{\psi}_\nu), \text{ since } \sum_{\nu\in\mathbb{Z}} (\varphi_\nu * \tilde{\psi}_\nu)\hat{}\,(\xi) = 1, \text{ by (29)},$$

and

$$\vec{g}\,(\vec{f}\,) = \sum_{\nu\in\mathbb{Z}} (\vec{g} * (\varphi_\nu * \tilde{\psi}_\nu), \vec{f}\,) = \sum_{\nu\in\mathbb{Z}} \Big((\vec{g} * \varphi_\nu), (\vec{f} * \psi_\nu)\Big)$$

$$= \sum_{\nu\in\mathbb{Z}} \sum_{Q\in Q_\nu} \int_Q \Big\langle A_Q A_Q^{-1}(\vec{g} * \varphi_\nu)(x), (\vec{f} * \psi_\nu)(x) \Big\rangle_{\mathcal{H}}\, dx$$

$$\leq \sum_{\nu\in\mathbb{Z}} \sum_{Q\in Q_\nu} \int_{\mathbb{R}^n} \|A_Q^{-1}(\vec{g}*\varphi_\nu)(x)\|_{\mathcal{H}} \, \|A_Q(\vec{f}*\psi_\nu)(x)\|_{\mathcal{H}} \chi_Q(x)\, dx,$$

by the self-adjointness of each A_Q and the Cauchy-Schwarz inequality. Using Hölder's inequality several times, we obtain

$$|\vec{g}\,(\vec{f}\,)| \leq \sum_{\nu\in\mathbb{Z}} 2^{\nu\alpha} \left\|(\vec{f}*\psi_\nu)\right\|_{L^p(\{A_Q\},\nu)} \cdot 2^{-\nu\alpha} \, \|(\vec{g}*\varphi_\nu)\|_{L^{p'}(\{A_Q^{-1}\},\nu)} \tag{42}$$

$$\leq \left\|\left\{2^{\nu\alpha}\|(\vec{f}*\psi_\nu)\|_{L^p(\{A_Q\},\nu)}\right\}_\nu\right\|_{l^q} \left\|\left\{2^{-\nu\alpha}\|(\vec{g}*\varphi_\nu)\|_{L^{p'}(\{A_Q^{-1}\},\nu)}\right\}_\nu\right\|_{l^{q'}}, \text{ if } 1<q<\infty,$$

and if $0<q\leq 1$, we bound (42) by

$$\left\|\left\{2^{\nu\alpha}\|(\vec{f}*\psi_\nu)\|_{L^p(\{A_Q\},\nu)}\right\}_\nu\right\|_{l^1} \left\|\left\{2^{-\nu\alpha}\|(\vec{g}*\varphi_\nu)\|_{L^{p'}(\{A_Q^{-1}\},\nu)}\right\}_\nu\right\|_{l^\infty}$$

$$\leq \left\|\left\{2^{\nu\alpha}\|(\vec{f}*\psi_\nu)\|_{L^p(\{A_Q\},\nu)}\right\}_\nu\right\|_{l^q} \left\|\left\{2^{-\nu\alpha}\|(\vec{g}*\varphi_\nu)\|_{L^{p'}(\{A_Q^{-1}\},\nu)}\right\}_\nu\right\|_{l^\infty}.$$

Combining cases and using the independence of $\dot{B}_p^{\alpha q}(\{A_Q\})$ and $\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ on the choice of the admissible kernel if $\{A_Q\}_Q$ is doubling (Corollary 4.9), we get

$$|\vec{g}\,(\vec{f}\,)| \leq \|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} \, \|\vec{g}\,\|_{\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})}.$$

Since $\mathcal{S}_0$ is dense in $\dot{B}_p^{\alpha q}(\{A_Q\})$, we get the above inequality for any $\vec{f}\in \dot{B}_p^{\alpha q}(\{A_Q\})$. Thus, $\vec{g}\in \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ belongs to $\left[\dot{B}_p^{\alpha q}(\{A_Q\})\right]^*$ and $\|\vec{g}\,\|_{oper} \leq \|\vec{g}\,\|_{\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})}$. ☐

Lemma 6.2. *Let $\alpha\in\mathbb{R}$, $1\leq p<\infty$, $0<q<\infty$ and $\{A_Q\}_Q$ be a doubling sequence of order p. Then*

$$\left[\dot{B}_p^{\alpha q}(\{A_Q\})\right]^* \subseteq \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}). \tag{43}$$

Proof. Let $l\in \left[\dot{B}_p^{\alpha q}(\{A_Q\})\right]^*$. We show that there exists $\vec{g}\in \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ such that $l(\vec{f}\,) = \vec{g}\,(\vec{f}\,) = (\vec{f},\vec{g}\,)$ for any $\vec{f}\in \dot{B}_p^{\alpha q}(\{A_Q\})$.

Case $1\leq q<\infty$. Take $\vec{f}\in \dot{B}_p^{\alpha q}(\{A_Q\})$, and for any $\nu\in\mathbb{Z}$ denote $\vec{f_\nu} = \vec{f}*\varphi_\nu$. Set T by $T(\{\vec{f_\nu}\}_\nu) = l(\vec{f}\,)$, so T is defined on a subspace of $l_q^\alpha(L^p(\{A_Q\},\nu))$. Since l isbounded, so is T:

$$|T(\{\vec{f_\nu}\}_\nu)| = |l(\vec{f}\,)| \leq c\,\|\vec{f}\,\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} = c\,\|\{\vec{f_\nu}\}_\nu\|_{l_q^\alpha(L^p(\{A_Q\},\nu))}.$$

Extend T, denote the extension $\tilde{T}$, onto all of $l_q^\alpha(L^p(\{A_Q\},\nu))$ (note: $q\geq 1$). Since $[l^q(X)]^* \approx l^{q'}(X^*)$ (cf. Section 3 or [1, Chapter 8]), we have $\left[l_q^\alpha(L^p(\{A_Q\},\nu))\right]^* \approx l_{q'}^{-\alpha}\left([L^p(\{A_Q\},\nu)]^*\right) \approx l_{q'}^{-\alpha}(L^{p'}(\{A_Q^{-1}\},\nu))$ by (37). Thus, there exists a vector-valued sequence $\{\vec{g_\nu}\}_{\nu\in\mathbb{Z}} \in l_{q'}^{-\alpha}(L^{p'}(\{A_Q^{-1}\},\nu))$ such that $\|\{\vec{g_\nu}\}_{\nu\in\mathbb{Z}}\|_{l_{q'}^{-\alpha}(L^{p'}(\{A_Q^{-1}\},\nu))} \leq \|l\|$ and for any $\vec{f}\in \dot{B}_p^{\alpha q}(\{A_Q\})$

$$l(\vec{f}\,) = \tilde{T}(\{\vec{f_\nu}\}) = T(\{\vec{f_\nu}\}) = \{\vec{g_\nu}\}(\{\vec{f_\nu}\}) = \sum_{\nu\in\mathbb{Z}} \int_{\mathbb{R}^n} \left\langle \vec{f_\nu}(x), \vec{g_\nu}(x)\right\rangle_{\mathcal{H}} dx$$

$$= \sum_{\nu\in\mathbb{Z}} \int_{\mathbb{R}^n} \langle (f * \varphi_\nu)(x), \vec{g}_\nu(x)\rangle_{\mathcal{H}}\, dx = \sum_{\nu\in\mathbb{Z}} \int_{\mathbb{R}^n} \left\langle \vec{f}(x), (\vec{g}_\nu * \tilde{\varphi}_\nu)(x)\right\rangle_{\mathcal{H}} dx.$$

Define $\vec{g}(x) = \sum_{\nu\in\mathbb{Z}} (\vec{g}_\nu * \tilde{\varphi}_\nu)(x)$. Then $l(\vec{f}) = (\vec{f}, \vec{g})$. Moreover, for any $\psi \in \mathcal{A}$ (by Corollary 4.9)

$$\|\vec{g}\,\|_{\dot{B}^{-\alpha q'}_{p'}(\{A_Q^{-1}\})} \approx \left\| \left\{ \left\| \sum_{\nu\in\mathbb{Z}} \vec{g}_\nu * \tilde{\varphi}_\nu * \psi_\mu \right\|_{L^{p'}(\{A_Q^{-1}\},\mu)} \right\}_\mu \right\|_{l^{-\alpha}_{q'}}$$

$$\le \left\| \left\{ \sum_{\nu\in\mathbb{Z}} \|\vec{g}_\nu * \tilde{\varphi}_\nu * \psi_\mu\|_{L^{p'}(\{A_Q^{-1}\},\mu)} \right\}_\mu \right\|_{l^{-\alpha}_{q'}}$$

$$= \left\| \left\{ \sum_{\nu=\mu-1}^{\mu+1} \|\vec{g}_\nu * \tilde{\varphi}_\nu * \psi_\mu\|_{L^{p'}(\{A_Q^{-1}\},\mu)} \right\}_\mu \right\|_{l^{-\alpha}_{q'}},$$

since $\operatorname{supp} \hat{\psi}_\mu \subseteq \{\xi :\ 2^{\mu-1} \le |\xi| \le 2^{\mu+1}\}$ and so $\tilde{\varphi}_\nu * \psi_\mu = 0$ if $|\mu-\nu| > 1$. Reindexing the inner sum, we get

$$\|\,\vec{g}\,\|^{q'}_{\dot{B}^{-\alpha q'}_{p'}(\{A_Q^{-1}\})} \le c \sum_{\mu\in\mathbb{Z}} 2^{-\mu\alpha q'} \sum_{j=-1}^{1} \|\vec{g}_{\mu+j} * \tilde{\varphi}_\mu * \psi_{\mu+j}\|^{q'}_{L^{p'}(\{A_Q^{-1}\},\mu)}.$$

Since $\{A_Q\}_Q$ is doubling and the sum on j is finite, we apply Lemma 5.6(ii) to get

$$\|\,\vec{g}\,\|_{\dot{B}^{-\alpha q'}_{p'}(\{A_Q^{-1}\})} \le c' \left\| \left\{ 2^{-\mu\alpha} \|\vec{g}_\mu\|_{L^{p'}(\{A_Q^{-1}\},\mu)} \right\}_\mu \right\|_{l^{q'}} \le \|l\|.$$

Case $0 < q < 1$. Take $\vec{f}$ with $(\vec{f})_i \in \mathcal{S}_0$. Since $\varphi \in \mathcal{S}_0$, for $\nu \in \mathbb{Z}$ by definition of convolution and then boundedness of l, we have

$$|(l * \varphi_\nu)(\vec{f})| = |l(\vec{f} * \tilde{\varphi}_\nu)| \le \|l\|\, \|\vec{f} * \tilde{\varphi}_\nu\|_{\dot{B}^{\alpha q}_p(\{A_Q\})}. \tag{44}$$

Note that each component of $l * \varphi_\nu$ is a $\mathcal{C}^\infty$-function and also $\|\vec{f} * \tilde{\varphi}_\nu\|_{\dot{B}^{\alpha q}_p(\{A_Q\})} \le 2^{\nu\alpha} \sum_{\mu=\nu-1}^{\nu+1} \|\vec{f} * \tilde{\varphi}_\nu\|_{L^p(\{A_Q\},\mu)} \le c\, 2^{\nu\alpha} \|\vec{f}\,\|_{L^p(\{A_Q\},\nu)}$ by Lemma 5.6(i). Substituting this estimate into (44), we get $|(l * \varphi_\nu)(\vec{f})| \le c\, 2^{\nu\alpha} \|l\|\, \|\vec{f}\,\|_{L^p(\{A_Q\},\nu)}$. By duality,

$$2^{-\nu\alpha} \|l * \varphi_\nu\|_{L^{p'}(\{A_Q^{-1}\},\nu)} = 2^{-\nu\alpha} \sup_{\vec{f}\in\mathcal{S}_0} \frac{|(l * \varphi_\nu)(\vec{f})|}{\|\vec{f}\,\|_{L^p(\{A_Q\},\nu)}} \le c\, \|l\|,$$

i.e., the functional $l * \varphi_\nu$ can be associated with a function $\vec{g}_\nu \in L^{p'}(\{A_Q^{-1}\}, \nu)$ such that $2^{-\nu\alpha} \cdot \|\vec{g}_\nu\|_{L^{p'}(\{A_Q^{-1}\},\nu)} \le c\, \|l\|$. Let $\vec{g} = \sum_{\nu\in\mathbb{Z}} \vec{g}_\nu * \theta_\nu$, where θ is as in the atomic decomposition theorem [3, Lemma 5.12] or [2, p.783], which implies $\hat{\vec{g}} = \sum_{\nu\in\mathbb{Z}} \hat{l}\hat{\varphi}_\nu\hat{\theta}_\nu = \hat{l}\cdot 1$ and so $\vec{g} = l$. Observe that $\vec{g} \in \dot{B}_{p'}^{-\alpha\infty}(\{A_Q^{-1}\})$:

$$\|\vec{g}\|_{\dot{B}_{p'}^{-\alpha\infty}(\{A_Q^{-1}\})} = \sup_{\nu\in\mathbb{Z}} 2^{-\nu\alpha}\|g*\varphi_\nu\|_{L^{p'}(\{A_Q^{-1}\},\nu)} \le c\,\|l\|.$$

Thus, the functional $l \in \left[\dot{B}_p^{\alpha q}(\{A_Q\})\right]^*$ can be associated with $\vec{g} \in \dot{B}_{p'}^{-\alpha\infty}(\{A_Q^{-1}\})$ and $l(\vec{f}) = (\vec{f},\vec{g})$. This completes the proof. □

Summarizing the results of this and the previous section we get the following embeddings of B-spaces:

Corollary 6.3. *Let W be a matrix weight and $\{A_Q\}_Q$ its reducing operators. Let $\alpha\in\mathbb{R}$, $0<q<\infty$ and $1\le p<\infty$. Then*

(45)
$$\left[\dot{B}_p^{\alpha q}(W)\right]^* \overset{(1)}{\subseteq} \left[\dot{B}_p^{\alpha q}(\{A_Q\})\right]^* \overset{(2)}{\subseteq} \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \overset{(3)}{\subseteq} \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{\#}\}) \overset{(4)}{\subseteq} \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}),$$

where

(1) *holds if W is doubling of order p,*
(2) *holds if W is doubling of order p,*
(3) *holds if $W\in A_p$, $1<p<\infty$,*
(4) *holds if $W^{-p'/p}$ is doubling of order p', $1<p<\infty$.*

Also,

(46)
$$\left[\dot{B}_p^{\alpha q}(W)\right]^* \overset{(1^*)}{\supseteq} \left[\dot{B}_p^{\alpha q}(\{A_Q\})\right]^* \overset{(2^*)}{\supseteq} \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \overset{(3^*)}{\supseteq} \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{\#}\}) \overset{(4^*)}{\supseteq} \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}),$$

where

(1^*) *holds if W satisfies any of (A1)-(A3),*
(2^*) *holds if W is doubling of order p,*
(3^*) *holds for any matrix weight W,*
(4^*) *holds if $W^{-p'/p}$ satisfies any of (A1)-(A3), $1<p<\infty$.*

In terms of a matrix weight W only, (45) and (46) are

$$\left[\dot{B}_p^{\alpha q}(W)\right]^* \subseteq \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}) \quad \text{if } W\in A_p,\ 1<p<\infty,$$

and

$$\left[\dot{B}_p^{\alpha q}(W)\right]^* \supseteq \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}) \quad \text{if } W,\ W^{-p'/p} \text{ satisfy any of (A1)-(A3)},\ 1<p<\infty.$$

In particular, if $W\in A_p$ (and so $W^{-p'/p}\in A_{p'}$), then $\left[\dot{B}_p^{\alpha q}(W)\right]^* \approx \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p})$, otherwise, ($W$ still satisfies any of (A1)-(A3), otherwise there may be a dependence on φ) $\left[\dot{B}_p^{\alpha q}(W)\right]^* \approx \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$, which completes the proof of Theorem A 1.

7. Duality of inhomogeneous Besov spaces $b_p^{\alpha q}(W)$ and $B_p^{\alpha q}(W)$

Recall that the main difference between homogeneous and inhomogeneous spaces is that instead of considering all dyadic cubes, we consider only the ones with side length $l(Q) \le 1$ and the properties of functions corresponding to $l(Q) = 1$ are slightly changed. We start with the sequence spaces. Recall the definition of the space $b_p^{\alpha q}(W)$.

Definition 7.1 (*Inhomogeneous matrix-weighted sequence Besov space $b_p^{\alpha q}(W)$*)**.** For $\alpha \in \mathbb{R}$, $0 < q \le \infty$, $1 \le p < \infty$ and W a matrix weight, the space $b_p^{\alpha q}(W)$ consists of all vector-valued sequences $\vec{s} = \{\vec{s}_Q\}_{l(Q)\le 1}$ such that

$$\|\vec{s}\|_{b_p^{\alpha q}(W)} = \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \vec{s}_Q \chi_Q \right\|_{L^p(W)} \right\}_{\nu \ge 0} \right\|_{l^q} < \infty.$$

Let $\mathcal{RS}^{(I)}$ be the collection of all sequences $\{A_Q\}_{l(Q)\le 1}$ of positive-definite operators on $\mathcal{H}$. Similar to the homogeneous case, we introduce the averaging space $b_p^{\alpha q}(\{A_Q\})$.

Definition 7.2. (*Inhomogeneous averaging matrix-weighted sequence Besov space $b_p^{\alpha q}(\{A_Q\})$*). For $\alpha \in \mathbb{R}$, $0 < q \le \infty$, $1 \le p \le \infty$ and $\{A_Q\}_{l(Q)\le 1} \in \mathcal{RS}^{(I)}$, let

$$b_p^{\alpha q}(\{A_Q\}) = \Big\{ \vec{s} = \{\vec{s}_Q\}_{l(Q)\le 1} :$$
$$\|\vec{s}\|_{b_p^{\alpha q}(\{A_Q\})} = \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \vec{s}_Q \chi_Q \right\|_{L^p(\{A_Q\},\nu)} \right\}_{\nu \ge 0} \right\|_{l^q} < \infty \Big\}.$$

Let $\vec{s} \in b_p^{\alpha q}(W)$. Define $\dot{\vec{s}} = \{\dot{\vec{s}}_Q\}_{Q \in \mathcal{D}}$ by setting $\dot{\vec{s}}_Q = \vec{s}_Q$ if $l(Q) \le 1$ and $\dot{\vec{s}}_Q = 0$ if $l(Q) > 1$. Note that $\vec{s}$ is the restriction of $\dot{\vec{s}}$ on $b_p^{\alpha q}(W)$. Applying (4), we get

$$\|\vec{s}\|_{b_p^{\alpha q}(W)} = \left\| \dot{\vec{s}} \right\|_{\dot{b}_p^{\alpha q}(W)} \approx \left\| \dot{\vec{s}} \right\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} = \|\vec{s}\|_{b_p^{\alpha q}(\{A_Q\})},$$

which proves the following proposition.

Proposition 7.3. *Let $\alpha \in \mathbb{R}$, $1 \le p < \infty$, $0 < q \le \infty$ and let W be a matrix weight with reducing operators $\{A_Q\}_Q$. Then*

$$b_p^{\alpha q}(W) \approx b_p^{\alpha q}(\{A_Q\}),$$

in the sense of the norm equivalence.

Note that it's enough to consider reducing operators A_Q generated by a matrix weight W only for dyadic cubes of side length $l(Q) \le 1$, i.e., $\{A_Q\}_{l(Q)\le 1}$.

Now we establish the duality.

Theorem A 3. *Let $\alpha \in \mathbb{R}$, $1 \le p < \infty$, $0 < q < \infty$ and let W be a matrix weight with reducing operators $\{A_Q\}_{l(Q)\le 1}$. Then*

$$\left[b_p^{\alpha q}(W) \right]^* \approx b_{p'}^{-\alpha q'}(\{A_Q^{-1}\}).$$

Moreover, if $W \in A_p$, $1 < p < \infty$, then

$$\left[b_p^{\alpha q}(W) \right]^* \approx b_{p'}^{-\alpha q'}(W^{-p'/p}).$$

To prove this theorem one can simply repeat the arguments from Section 3 with proper adjustments (for example, consider sums on ν taken only over $\nu \geq 0$). However, we would like to give a simple proof for the embedding

$$\left[b_p^{\alpha q}(\{A_Q\})\right]^* \subseteq b_{p'}^{-\alpha q'}(\{A_Q^{-1}\}).$$

Proof. Let $l \in \left[b_p^{\alpha q}(\{A_Q\})\right]^*$. Let P be the projection from $\dot{b}_p^{\alpha q}(\{A_Q\})$ to $b_p^{\alpha q}(\{A_Q\})$ defined by restricting a sequence $\{\vec{s}_Q\}_{Q\in\mathcal{D}}$ to $\{\vec{s}_Q\}_{l(Q)\leq 1}$. Set $\tilde{l}$ by $\tilde{l}(\vec{s}) = l(P\vec{s})$ for each $\vec{s} \in \dot{b}_p^{\alpha q}(\{A_Q\})$. Then $\tilde{l} \in \left[\dot{b}_p^{\alpha q}(\{A_Q\})\right]^*$, since

$$|\tilde{l}(\vec{s})| = |l(P\vec{s})| \leq \|l\|\, \|P\vec{s}\|_{b_p^{\alpha q}(\{A_Q\})} \leq \|l\|\, \|\vec{s}\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}.$$

Then by Lemma 3.3 (or, equivalently, by (17)) $\tilde{l}$ is represented by $\tilde{\vec{t}} \in \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ such that $\tilde{l}(\vec{s}) = (\vec{s}, \tilde{\vec{t}})$ and $\|\tilde{\vec{t}}\|_{\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} \leq \|\tilde{l}\| \leq \|l\|$. Let $\vec{t} = P\tilde{\vec{t}}$. For $\vec{s} \in b_p^{\alpha q}(\{A_Q\})$ define $\dot{\vec{s}} \in \dot{b}_p^{\alpha q}(\{A_Q\})$ as above (thus, $P\dot{\vec{s}} = \vec{s}$). Then

$$l(\vec{s}) = \tilde{l}(\dot{\vec{s}}) = (\dot{\vec{s}}, \tilde{\vec{t}}) = \sum_{l(Q)\leq 1} \dot{\vec{s}}_Q\, \tilde{\vec{t}}_Q + \sum_{l(Q)>1} \dot{\vec{s}}_Q\, \tilde{\vec{t}}_Q = \sum_{l(Q)\leq 1} \dot{\vec{s}}_Q\, \tilde{\vec{t}}_Q = (\vec{s}, \vec{t}),$$

since $\dot{\vec{s}}_Q = 0$ for $l(Q) > 1$. Moreover, $\|\vec{t}\|_{b_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} \leq \|\tilde{\vec{t}}\|_{\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} \leq \|l\|$. □

Consider a class of functions $\mathcal{A}^{(I)}$ with properties similar to the ones of an admissible kernel: $\Phi \in \mathcal{A}^{(I)}$ if $\Phi \in \mathcal{S}(\mathbb{R}^n)$, supp $\hat{\Phi} \subseteq \{\xi \in \mathbb{R}^n : \ |\xi| \leq 2\}$ and $|\hat{\Phi}(\xi)| \geq c > 0$ if $|\xi| \leq \dfrac{5}{3}$. Recall the inhomogeneous space $B_p^{\alpha q}(W)$ from [5].

Definition 7.4 (*Inhomogeneous matrix-weighted Besov space* $B_p^{\alpha q}(W)$)**.** For $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$, W a matrix weight, $\varphi \in \mathcal{A}$ and $\Phi \in \mathcal{A}^{(I)}$, we define the Besov space $B_p^{\alpha q}(W)$ to be the collection of all vector-valued distributions $\vec{f} = (f_1, ..., f_m)^T$ with $f_i \in \mathcal{S}'(\mathbb{R}^n), 1 \leq i \leq m$ such that

$$\left\|\vec{f}\right\|_{B_p^{\alpha q}(W)} = \left\|\Phi * \vec{f}\right\|_{L^p(W)} + \left\|\left\{2^{\nu\alpha}\left\|\varphi_\nu * \vec{f}\right\|_{L^p(W)}\right\}_{\nu\geq 1}\right\|_{l^q} < \infty.$$

Analogously, we introduce the averaging space $B_p^{\alpha q}(\{A_Q\})$.

Definition 7.5 (*Averaging matrix-weighted Besov space* $B_p^{\alpha q}(\{A_Q\})$)**.** For $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p \leq \infty$, $\varphi \in \mathcal{A}$, $\Phi \in \mathcal{A}^{(I)}$ and $\{A_Q\}_{l(Q)\leq 1} \in \mathcal{RS}^{(I)}$, let

$$B_p^{\alpha q}(\{A_Q\}) = \Big\{\vec{f} = (f_1, ..., f_m)^T \text{ with } f_i \in \mathcal{S}'(\mathbb{R}^n), 1 \leq i \leq m :$$

$$\left\|\vec{f}\right\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} = \left\|\Phi * \vec{f}\right\|_{L^p(\{A_Q\},0)} + \left\|\left\{2^{\nu\alpha}\left\|\varphi_\nu * \vec{f}\right\|_{L^p(\{A_Q\},\nu)}\right\}_{\nu\geq 1}\right\|_{l^q} < \infty\Big\}.$$

Now the remaining results of Sections 4, 5, 6 transfer easily to the inhomogeneous Besov spaces by using the properties established in Section 12 of [5], such as replacing a family $\{\varphi_\nu\}_{\nu\in\mathbb{Z}}$ with $\{\varphi_\nu\}_{\nu\in\mathbb{N}} \cup \Phi$; observing that $\Phi * \vec{f} \in E_0$ and summing over $\nu \geq 0$ (or $l(Q) \leq 1$) in all sums. In particular, we get

Theorem 7.6. *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$ and let $\{A_Q\}_{l(Q)\leq 1}$ be a doubling sequence (of order p). Then for $\vec{s}_Q(\vec{f}) = \left\langle \vec{f}, \varphi_Q \right\rangle$*

$$\left\| \vec{f} \right\|_{B_p^{\alpha q}(\{A_Q\})} \approx \left\| \left\{ \vec{s}_Q \left(\vec{f} \right) \right\}_{l(Q)\leq 1} \right\|_{b_p^{\alpha q}(\{A_Q\})}.$$

Corollary 7.7. *The spaces $B_p^{\alpha q}(\{A_Q\})$, $B_p^{\alpha q}(\{A_Q^{-1}\})$ and $B_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ are independent of the choice of the pair of admissible kernels (φ, Φ), if $\{A_Q\}_{l(Q)\leq 1}$ is doubling (of order p), $1 \leq p < \infty$, $\alpha \in \mathbb{R}$, $0 < q \leq \infty$.*

Lemma 7.8. *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$ and $1 < p < \infty$. If W satisfies any of (A1)-(A3) and $\{A_Q\}_{l(Q)\leq 1}$ is a sequence of reducing operators generated by W, then*

$$B_p^{\alpha q}(W) \approx B_p^{\alpha q}(\{A_Q\}).$$

Theorem A 4. *Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \leq p < \infty$ and let $\{A_Q\}_{l(Q)\leq 1}$ be reducing operators of a matrix weight W. If W satisfies any of (A1)-(A3), then*

$$\left[B_p^{\alpha q}(W) \right]^* \approx B_{p'}^{-\alpha q'}(\{A_Q^{-1}\}).$$

If $W \in A_p$, $1 < p < \infty$, then

$$\left[B_p^{\alpha q}(W) \right]^* \approx B_{p'}^{-\alpha q'}(W^{-p'/p}).$$

8. Acknowledgements

The author would like ot thank you Prof. Michael Frazier for comments and suggesions while examining the paper.

References

[1] R.Edwards, *Functional Analysis: Theory and Applications,* Holt, Rinehart and Winston, Inc., 1965.

[2] M.Frazier and B.Jawerth, *Decomposition of Besov Spaces*, Indiana Univ. Math. J. **34** (1985), 777-799.

[3] M.Frazier, B.Jawerth and G.Weiss, *Littlewood-Paley Theory and Study of Function Spaces*, CBMS Regional Conference Series in Mathematics **79**, AMS, Providence, RI, 1991.

[4] F.Nazarov and S.Treil, *The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis*, Algebra i Analiz, **8** (1996), no. 5, 32-162 (in Russian).

[5] S.Roudenko, *Matrix-weighted Besov spaces,* Trans. Amer. Math. Soc. 355 (2003), 273-314.

[6] S.Roudenko, *The Theory of Function Spaces with Matrix Weights,* PhD thesis, Mich. State Univ., 2002.

[7] E.Stein, *Harmonic Analysis,* Princeton University Press, 1993.

[8] H.Triebel, *Theory of function spaces*, Monographs in Math., vol. 78, Birkhäuser Verlag, Basel, 1983.

[9] S.Treil and A.Volberg, *Wavelets and the angle between past and future*, J. Funct. Anal. **143** (1997), no.2, 269-308.

[10] A.Volberg, *Matrix A_p weights via S-functions*, J. Amer. Math. Soc. **10** (1997), no.2, 445-466.

Mathematics Department, Duke University, Durham, North Carolina 27708
E-mail address: `svetlana@math.duke.edu`