Practice Final Solutions

1. Find the solution to the initial value problem

$$y' = e^x y^2 + 2xy^2, \qquad y(0) = 1.$$

Your answer should be in explicit form: $y = f(x)$.

Solution: Factoring out y^2 on the right side, we can write the equation as

$$\frac{dy}{dx} = y^2(e^x + 2x), \quad \text{or} \quad \frac{dy}{y^2} = (e^x + 2x)\,dx,$$

which shows that this equation is separable. Integrating both sides using

$$\int \frac{1}{y^2}\,dy = \int y^{-2}\,dy = y^{-1}/(-1) = -1/y$$

gives

$$\frac{-1}{y} = \int (e^x + 2x)\,dx = e^x + x^2 + c.$$

Therefore, $y = \dfrac{-1}{e^x + x^2 + c}$. The initial condition $y(0) = 1$ gives $1 = \dfrac{-1}{e^0 + 0^2 + c} = \dfrac{-1}{1 + c}$, which yields $-1 = c + 1$, hence $c = -2$. Thus, the particular solution is

$$y = \frac{-1}{e^x + x^2 - 2}.$$

2. Find the general solution to $\dfrac{dy}{dx} = \dfrac{-3xy^2 - \cos x}{3x^2y + \sin y}$. You may leave your answer in implicit form.

Solution: The equation is the same as $(3x^2y + \sin y)dy = -(3xy^2 + \cos x)dx$, or $(3xy^2 + \cos x)dx + (3x^2y + \sin y)dy = 0$. Let $M = 3xy^2 + \cos x$ and $N = 3x^2y + \sin y$. Then $M_y = 6xy$ and $N_x = 6xy$. Since $M_y = N_x$, this equation is exact. Therefore there is a function $\psi(x, y)$ satisfying $\psi_x = M$ and $\psi_y = N$. To find ψ, we first solve

$$\psi_x = M = 3xy^2 + \cos x \quad \text{to get} \quad \psi = \frac{3}{2}x^2y^2 + \sin x + h(y).$$

Then all we need to know is h. For this, we compute

$$\psi_y = 3x^2y + h'(y) = N = 3x^2y + \sin y.$$

Hence $h'(y) = \sin y$. So $h(y) = -\cos y$. Substituting h into the formula for ψ above,

$$\psi = \frac{3}{2}x^2y^2 + \sin x - \cos y.$$

Hence the solution is $\psi = C$, or

$$\frac{3}{2}x^2y^2 + \sin x - \cos y = C.$$

Attention: Review all 4 methods to solve first order differential equations from Chapter 2.

3. A spring is stretched a distance of $\frac{32}{9}$ *ft* by a mass m which has weight $mg = 2$ *lbs*, where $g = 32$ *ft/sec*^{2} is the acceleration due to gravity. The spring is at rest until time $t = 0$, when an external force $F(t) = 3\sin 3t$ *lbs* is applied to the spring. Assuming no damping, find the position $u(t)$ of the spring at time t, where $u = 0$ corresponds to the rest position of the spring.

Solution: Because the weight is 2 *lbs*, the mass satisfies $2 = mg = m \cdot 32$, or $m = \frac{2}{32} = \frac{1}{16}$ *lbs-sec*^{2}*/ft*. The spring constant k is determined by the equation $mg = kL$, where L is the equilibrium distance that the spring is stretched by the mass m. In this case, $L = 32/9$ *ft*, so we have $2 = k\frac{32}{9}$ so $k = \frac{18}{32} = \frac{9}{16}$ *lbs/ft*. The general equation of motion for an undamped, forced spring is $mu'' + ku = F(t)$, where F is the forcing function. Here we obtain $\frac{1}{16}u'' + \frac{9}{16}u = 3\sin 3t$, or

$$u'' + 9u = 16 \cdot 3\sin 3t = 48\sin 3t.$$

Since the spring is at rest at time 0, we have the initial conditions $u(0) = 0$ and $u'(0) = 0$.

To solve, we first consider the homogeneous equation $u'' + 9u = 0$. The associated characteristic equation is $r^2 + 9 = 0$, or $r^2 = -9$, or $r = \pm 3i$. Therefore the homogeneous solution is $u_h = C_1\cos 3t + C_2\sin 3t$.

To find a particular solution u_p of $u'' + 9u = 48\sin 3t$, we first guess a solution of the form $u_p = A\cos 3t + B\sin 3t$. However, the terms $A\cos 3t$ and $B\sin 3t$ are in common with the terms $C_1\cos 3t$ and $C_2\sin 3t$ of u_h, so we have to modify the form of u_p. So we take

$$u_p = At\cos 3t + Bt\sin 3t.$$

Differentiating, we get

$$u_p{}' = A\cos 3t - 3At\sin 3t + B\sin 3t + 3Bt\cos 3t.$$

Differentiating again, we get

$$\begin{aligned} u_p{}'' &= -3A\sin 3t - 3A\sin 3t - 9At\cos 3t + 3B\cos 3t + 3B\cos 3t - 9Bt\sin 3t \\ &= -6A\sin 3t - 9At\cos 3t + 6B\cos 3t - 9Bt\sin 3t. \end{aligned}$$

Substituting into the equation $u'' + 9u = 48\sin 3t$, we get

$$-6A\sin 3t - 9At\cos 3t + 6B\cos 3t - 9Bt\sin 3t + 9At\cos 3t + 9Bt\sin 3t = 48\sin 3t.$$

The terms involving $t\cos 3t$ and $t\sin 3t$ cancel out, leaving

$$-6A\sin 3t + 6B\cos 3t = 48\sin 3t.$$

Hence, $A = -8$ and $B = 0$, so $u_p = -8t\cos 3t$.

Thus, the general solution is

$$u(t) = u_h(t) + u_p(t) = C_1\cos 3t + C_2\sin 3t - 8t\cos 3t.$$

Substituting the initial condition $u(0) = 0$ gives $0 = C_1$. So $u(t) = C_2\sin 3t - 8t\cos 3t$. Hence

$$u'(t) = 3C_2\cos 3t - 8\cos 3t + 24t\sin 3t.$$

Substituting the initial condition $u'(0) = 0$ gives $0 = 3C_2 - 8$, hence $3C_2 = 8$ or $C_2 = 8/3$. Therefore,

$$u(t) = \frac{8}{3}\sin 3t - 8t\cos 3t.$$

4. Find the first 3 non-zero terms in the power series expansion at the origin for each of the two linearly indendent solutions of

$$y'' + 2xy' - 3y = 0.$$

Solution: Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Hence,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Substituting these series for y, y', and y'' into the equation $y'' + 2xy' - 3y = 0$, we get

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x\sum_{n=1}^{\infty} n a_n x^{n-1} - 3\sum_{n=0}^{\infty} a_n x^n = 0.$$

We re-index the first sum by setting $m = n - 2$, or equivalently, $n = m + 2$, to obtain $\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m$ (note that as n starts at 2, m starts at 0). For the second term, we multiply the factor $2x$ inside the sum to get $\sum_{n=0}^{\infty} 2na_nx^n$, and then we replace n by m to obtain $\sum_{m=0}^{\infty} 2ma_mx^m$. In the third series, we just multiply the factor of (-3) inside, and replace n by m to get $\sum_{m=0}^{\infty} -3a_mx^m$. Making these substitutions, we get

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m + \sum_{m=0}^{\infty} 2ma_mx^m + \sum_{m=0}^{\infty} -3a_mx^m = 0.$$

Combining these terms into one sum and factoring out x^m produces

$$\sum_{m=0}^{\infty}[(m+2)(m+1)a_{m+2} + 2ma_m - 3a_m]x^m = 0.$$

By uniqueness for series, this forces

$$(m+2)(m+1)a_{m+2} + 2ma_m - 3a_m = 0$$

for every $m = 0, 1, 2, 3, ...$ Simplifying, we get $(m+2)(m+1)a_{m+2} = 3a_m - 2ma_m = (3-2m)a_m$, or

$$a_{m+2} = \frac{3-2m}{(m+2)(m+1)}a_m,$$

which is the recurrence relation.

We take a_0 and a_1 to be arbitrary. Using the recurrence relation $a_{m+2} = \dfrac{3-2m}{(m+2)(m+1)}a_m$ with $m = 0$ gives $a_2 = \dfrac{3}{2}a_0$. With $m = 1$ we get $a_3 = \dfrac{1}{6}a_1$. With $m = 2$ we obtain $a_4 = \dfrac{-1}{12}a_2 = \dfrac{-1}{12}\cdot\dfrac{3}{2}a_2 = \dfrac{-1}{8}a_0$. Taking $m = 3$, we obtain $a_5 = \dfrac{-3}{20}a_3 = \dfrac{-3}{20}\cdot\dfrac{1}{6}a_1 = \dfrac{-1}{40}a_1$.

Thus, we obtain

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + ... \\ &= a_0 + a_1x + \frac{3}{2}a_0x^2 + \frac{1}{6}a_1x^3 - \frac{1}{8}a_0x^4 - \frac{1}{40}a_1x^5 + ... \\ &= a_0[1 + \frac{3}{2}x^2 - \frac{1}{8}x^4 + ...] + a_1[x + \frac{1}{6}x^3 - \frac{1}{40}x^5 + ...]. \end{aligned}$$

Therefore,

$$y_1 = 1 + \frac{3}{2}x^2 - \frac{1}{8}x^4 + ... \quad \text{and} \quad y_2 = x + \frac{1}{6}x^3 - \frac{1}{40}x^5 + ... $$

5. Find the singular points of the equation

$$(x-1)^2(x-3)^2y'' + 8(x-1)y' + (x-3)y = 0.$$

<u>Solution:</u> Observe that $P(x) = (x-1)^2\,(x-3)^2$, equating it to zero we obtain $(x-1)^2\,(x-3)^2 = 0$, or $x = 1$ and $x = 3$. Thus, the singular points for this equation are $x = 1, 3$. At these points the power series solution (as the Taylor expansion) does not exist. All other real numbers $\mathbb{R} \setminus \{1, 3\}$ are ordinary points.

6. (a) Find a first order linear system of differential equations which is equivalent to the second order equation $y'' + 2y' - 15y = 0$. If this system is written in the form $\mathbf{x}' = A\mathbf{x}$, find the matrix A.

<u>Solution:</u> Let $x_1 = y$ and $x_2 = y'$. Then $x_1{}' = y' = x_2$ and $x_2{}' = y'' = 15y - 2y' = 15x_1 - 2x_2$, using the equation $y'' + 2y' - 15y = 0$. So the system is $\begin{array}{c} x_1{}' = 0x_1 + x_2 \\ x_2{}' = 15x_1 - 2x_2 \end{array}$. In matrix form, the equation is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{pmatrix} 0 & 1 \\ 15 & -2 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, or $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix} 0 & 1 \\ 15 & -2 \end{pmatrix}$.

(b) Solve the system found in part (a). Use this to find the general solution y of $y'' + 2y' - 15y = 0$.

Solution Since $A-\lambda I = \begin{pmatrix} -\lambda & 1 \\ 15 & -2-\lambda \end{pmatrix}$, we get det $(A-\lambda I) = (-\lambda)(-2-\lambda)-15 = 2\lambda+\lambda^2-15 = \lambda^2 + 2\lambda - 15 = (\lambda+5)(\lambda-3) = 0$ when $\lambda = -5$ and $\lambda = 3$.

To find the eigenvector for $\lambda = -5$, we solve $(A-(-5)I)v = 0$, or $\begin{pmatrix} 5 & 1 \\ 15 & 3 \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which gives $5a + b = 0$. We take $a = 1$ and $b = -5$, so $v = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ and $\mathbf{x_1} = e^{-5t} \begin{bmatrix} 1 \\ -5 \end{bmatrix}$.

For $\lambda = 3$, we solve $(A - 3I)v = 0$, or $\begin{pmatrix} -3 & 1 \\ 15 & -5 \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which gives $-3a + b = 0$, or $b = 3a$. We take $a = 1$ and $b = 3$, so $v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{x_2} = e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Thus the general solution is $\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2} = c_1 e^{-5t} \begin{bmatrix} 1 \\ -5 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Hence the general solution of $y'' + 2y' - 15y = 0$ is (by part A) $y = x_1 = C_1 e^{-5t} + C_2 e^{3t}$.

(c) Find the general solution y of the equation $y'' + 2y' - 15y = 0$ and compare with the answer found in part (b).

Solution: Trying for a solution of the form $y = e^{rt}$, we get the characteristic equation $r^2+2r-15 = 0$, or $(r+5)(r-3) = 0$, hence $r = -5$ and $r = 3$ are the solutions. Therefore the general solution is $y = C_1 e^{-5t} + C_2 e^{3t}$, which agrees with the answer in part B.

(d) Let W_1 be the Wronskian of the solutions to the system in part (a). Let W_2 be the Wronskian of two linearly independent solutions to $y'' + 2y' - 15y = 0$. Compute W_1 and W_2 and compare them.

Solution: We have $W_1 = \det [\mathbf{x_1}\ \mathbf{x_2}] = \det \begin{pmatrix} e^{-5t} & e^{3t} \\ -5e^{-5t} & 3e^{3t} \end{pmatrix} = 3e^{-2t} - (-5)e^{-2t} = 8e^{-2t}$. On the other hand, letting $y_1 = e^{-5t}$ and $y_2 = e^{3t}$ be the linearly independent solutions of $y'' + 2y' - 15y = 0$ found in part B, we get $W_2 = \det \begin{pmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{pmatrix} = \det \begin{pmatrix} e^{-5t} & e^{3t} \\ -5e^{-5t} & 3e^{3t} \end{pmatrix} = 3e^{-2t} - (-5)e^{-2t} = 8e^{-2t}$. So $W_1 = W_2$.

7. (a) Find 2 linearly independent real-valued solutions to the system

$$\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} \mathbf{x}\,.$$

Solution: For $A = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$, we have $A - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix}$. Hence $\det(A - \lambda I) = (3-\lambda)^2 - (-1) = 9 - 6\lambda + \lambda^2 + 1 = \lambda^2 - 6\lambda + 10 = 0$ when

$$\lambda = \frac{-(-6) \pm \sqrt{36 - 4 \cdot 10}}{2} = \frac{6 \pm \sqrt{-4}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i.$$

Taking one of the eigenvalues, say $\lambda = 3 + i$, we find the corresponding eigenvector by solving $(A - (3+i)I)v = 0$, or $\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The first component gives the equation $-ia + b = 0$ (if you multiply this equation by $-i$ you get the equation $-a - ib = 0$ corresponding to the second component). So we can choose a solution $a = 1, b = i$; that is, $v = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Therefore a solution of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = e^{(3+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^{3t} e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^{3t}(\cos t + i \sin t) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^{3t}\begin{bmatrix} \cos t + i\sin t \\ -\sin t + i\cos t \end{bmatrix} = e^{3t}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + ie^{3t}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

The real and imaginary parts of this solution must also be solutions, so we get the two real-valued solutions:

$$\mathbf{x_1} = e^{3t}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \text{ and } \mathbf{x_2} = e^{3t}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

The graph will be an unstable spiral (since $\mathrm{Re}\lambda = 3 > 0$) and the direction of arrows will be clockwise.

(b) Select one of the solutions you obtained in part (a) and check that it in fact satisfies the system $\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\mathbf{x}$.

Solution: For example, taking $\mathbf{x_1} = \begin{bmatrix} e^{3t}\cos t \\ -e^{3t}\sin t \end{bmatrix}$, we have

$$\mathbf{x_1}' = \begin{bmatrix} 3e^{3t}\cos t - e^{3t}\sin t \\ -3e^{3t}\sin t - e^{3t}\cos t \end{bmatrix},$$

and

$$A\mathbf{x_1} = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\begin{bmatrix} e^{3t}\cos t \\ -e^{3t}\sin t \end{bmatrix} = \begin{bmatrix} 3e^{3t}\cos t - e^{3t}\sin t \\ -e^{3t}\cos t - 3e^{3t}\sin t \end{bmatrix} = \begin{bmatrix} 3e^{3t}\cos t - e^{3t}\sin t \\ -3e^{3t}\sin t - e^{3t}\cos t \end{bmatrix}.$$

So $A\mathbf{x_1} = \mathbf{x_1}$, so $\mathbf{x_1}$ is a solution.

8. Find the solution of the initial value problem:

$$\mathbf{x}' = \begin{pmatrix} 5 & 1 \\ 3 & 5 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Solution Find eigenvalues $\det(A-\lambda I) = 0$, you get $(\lambda-5)^2 - 3 = 0$, and so $\lambda_1 = 5+\sqrt{3}$ and $\lambda_2 = 5-\sqrt{3}$. Note that both eigenvectors are positive and real, so the sketch of solution will be an **unstable node**.

Now eigenvectors: for λ_1, you get a system of equations:

$-\sqrt{3}a + b = 0$

$3a - \sqrt{3}b = 0$,

which are similar and, for example, $\mathbf{v}_1 = (1, \sqrt{3})^T$;

for for λ_2, you get a system of equations:

$\sqrt{3}a + b = 0$

$3a + \sqrt{3}b = 0$,

which are similar and $\mathbf{v}_2 = (1, -\sqrt{3})^T$.

General solution:

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}e^{(5+\sqrt{3})t} + c_2\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}e^{(5-\sqrt{3})t}$$

Now, the initial value problem:

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix},$$

so we obtain

$$c_1 + c_2 = 2$$
$$c_1\sqrt{3} - c_2\sqrt{3} = -1,$$

solving it we get $c_1 = 1 - \frac{1}{2\sqrt{3}}$ and $c_1 = 1 + \frac{1}{2\sqrt{3}}$.

Hence, the solution is

$$\mathbf{x}(t) = (1 - \frac{1}{2\sqrt{3}}) \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} e^{(5+\sqrt{3})t} + (1 + \frac{1}{2\sqrt{3}}) \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} e^{(5-\sqrt{3})t}.$$

9. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix} \mathbf{x}.$$

Solution: Find eigenvalues first: $\det(A - \lambda I) = 0$, which gives $(3-\lambda)[(2-\lambda)(2-\lambda)-1] = -(3-\lambda)^2(\lambda-1)$. Hence, $\lambda = 3$ is an eigenvalue of multiplicity 2 and $\lambda = 1$ is an eigenvalue of multiplicity 1.

To find the eigenvector corresponding to $\lambda = 1$, we solve $(A - 1 \cdot I)\mathbf{v} = 0$, here $\mathbf{v} = (a, b, c)^T$. The first row gives $a = 0$. Given this, the second and third rows give $b + c = 0$. So $c = -b$, and one choice of eigenvector is $\mathbf{v_1} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ So $\mathbf{x_1} = e^t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

For $\lambda = 3$, solve $(A - 3I)\mathbf{v} = 0$. This gives only one equation $2a - b + c = 0$ (the third equation is equivalent to this one). Hence, we can take a and b to be arbitrary, as long as $c = -2a + b$. We obtain $\mathbf{v} = \begin{pmatrix} a \\ b \\ -2a + b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. So $\lambda = 3$ has two linearly independent eigenvectors (why linearly independent?, can you show?) $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Therefore, we obtain $\mathbf{x_2} = e^{3t} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\mathbf{x_3} = e^{3t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

The general solution is $\mathbf{x}(t) = c_1\mathbf{x_1} + c_2\mathbf{x_2} + c_3\mathbf{x_3}$.

10. Consider the differential equation

$$(\star) \qquad (x^2 - 1)y'' - 6y = 0.$$

(a) Find the recurrence relation for the coefficients in the series expansion of y at the origin.

Solution: Let $y = \sum_{n=0}^{\infty} a_n x^n$. Hence,

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \textit{and} \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

We multiply out the equation $(x^2 - 1)y'' - 6y = 0$ to get $x^2 y'' - y'' - 6y = 0$. Substituting the series for y and y'', we get

$$x^2 \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - 6 \sum_{n=0}^{\infty} a_n x^n = 0.$$

For the first series, we multiply the factor x^2 inside the sum to get $\sum_{n=0}^{\infty} n(n-1) a_n x^n$ and then replace n by m to get $\sum_{m=0}^{\infty} m(m-1) a_m x^m$. In the second series, we note that we can start the sum at $n = 2$ since the terms for $n = 0$ and $n = 1$ are 0, obtaining $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$, and we re-index the sum by setting $m = n-2$ or equivalently $n = m+2$, to obtain $\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m$ (note that as n starts at 2, m starts at 0). In the third series, we just replace n by m to get $\sum_{m=0}^{\infty} a_m x^m$. Making these substitutions, we get

$$\sum_{m=0}^{\infty} m(m-1)a_m x^m - \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - 6\sum_{m=0}^{\infty} a_m x^m = 0.$$

Combining these terms into one sum and factoring out x^m gives

$$\sum_{m=0}^{\infty}[m(m-1)a_m - (m+2)(m+1)a_{m+2} - 6a_m]x^m = 0.$$

By uniqueness for series, this forces

$$m(m-1)a_m - (m+2)(m+1)a_{m+2} - 6a_m = 0$$

for every $m = 0, 1, 2, 3, \ldots$. Simplifying, we get $[m(m-1)-6]a_m = (m+2)(m+1)a_{m+2}$ or $(m^2-m-6)a_m = (m+2)(m+1)a_{m+2}$. Factoring, we have $(m-3)(m+2)a_m = (m+2)(m+1)a_{m+2}$. Since $m \neq -2$, we can cancel the term $m+2$ from both sides to get $(m-3)a_m = (m+1)a_{m+2}$. Finally, solving for a_{m+2}, we get $a_{m+2} = \frac{m-3}{m+1}a_m$.

(b) Find the solution y of $(\star)$ that satisfies $y(0) = 0$ and $y'(0) = 1$.

Solution: We have $a_0 = y(0) = 0$, hence from the recurrence relation $a_{m+2} = \frac{m-3}{m+1}a_m$ we get $a_2 = \frac{-3}{1}a_0 = 0$, then $a_4 = 0, a_6 = 0$, etc., so $a_n = 0$ for all even values of n. Also, we have $a_1 = y'(0) = 1$, so from the recurrence relation with $m = 1$ we get $a_3 = \frac{-2}{2}a_1 = -1 \cdot 1 = -1$. Then with $m = 3$ we get $a_5 = \frac{0}{4}a_3 = 0$. Then with $m = 5$ we get $a_7 = \frac{2}{6}a_3 = \frac{1}{3}0 = 0$, and similarly $a_9 = 0, a_{11} = 0$, etc. So all a_n are 0 except when $n = 1$ and $n = 3$. So $y = a_1 x + a_3 x^3 = x - x^3$, since $a_1 = 1$ and $a_3 = -1$.

(c) Find a lower bound for the radius of convergence of the series around the origin for any solution of $(\star)$. Give reasons for your answer.

Solution: Dividing the equation by (x^2-1), we have $y'' - \frac{6}{x^2-1}y = 0$. The only singularities of the coefficient function $\frac{6}{x^2-1}$ are when $x^2 - 1 = 0$, or $x = \pm 1$. The distance of ± 1 from the base point $x = 0$ is 1, so the lower bound for the radius of convergence ρ of the series for the solution y at the origin is 1; that is, we have $\rho \geq 1$.

11. Find the first 3 non-zero terms in the series solution at $x_0 = 0$ for each of two linearly independent soutions of the differential equation

$$y'' - 2xy' - 2y = 0.$$

Solution: Let $y = \sum_{n=0}^{\infty} a_n x^n$. Hence,

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}.$$

Substituting these series for y, y', and y'' into the equation $y'' - 2xy' - 2y = 0$, we get

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} na_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

For the first series, we note that we can start the sum at $n = 2$ since the terms for $n = 0$ and $n = 1$ are 0, obtaining $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. Then we re-index the sum by setting $m = n - 2$, or equivalently, $n = m + 2$, to obtain $\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m$ (note that as n starts at 2, m starts at 0). For the second term, we multiply the factor $-2x$ inside the sum to get $\sum_{n=0}^{\infty} -2na_n x^n$, and then we replace n by m to obtain $\sum_{m=0}^{\infty} -2ma_m x^m$. In the third series, we just multiply the factor of -2 inside, and replace n by m to get $\sum_{m=0}^{\infty} -2a_m x^m$. Making these substitutions, we get

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m + \sum_{m=0}^{\infty} -2ma_m x^m + \sum_{m=0}^{\infty} -2a_m x^m = 0.$$

Combining these terms into one sum and factoring out x^m gives $\sum_{m=0}^{\infty}[(m+2)(m+1)a_{m+2} - 2ma_m - 2a_m]x^m = 0$. By uniqueness for series, this forces

$$(m+2)(m+1)a_{m+2} - 2ma_m - 2a_m = 0$$

for every $m = 0, 1, 2, 3, ...$ Simplifying, we get $(m+2)\,(m+1)\,a_{m+2} = 2m\,a_m + 2a_m = 2(m+1)a_m$. Since $m \neq -1$, we can cancel the term $m+1$ from both sides to get $(m+2)a_{m+2} = 2a_m$, or $a_{m+2} = \frac{2}{m+2}a_m$.

We take a_0 and a_1 to be arbitrary. Using the recurrence relation $a_{m+2} = \frac{2}{m+2}a_m$ with $m = 0$ gives $a_2 = \frac{2}{2}a_0 = a_0$. With $m = 1$ we get $a_3 = \frac{2}{3}a_1$. With $m = 2$ we obtain $a_4 = \frac{2}{4}a_2 = \frac{1}{2}a_2 = \frac{1}{2}a_0$. Taking $m = 3$, we obtain $a_5 = \frac{2}{5}a_3 = \frac{2}{5}\frac{2}{3}a_1 = \frac{4}{15}a_1$.

Thus, we obtain

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + ... \\ &= a_0 + a_1x + a_0x^2 + \frac{2}{3}a_1x^3 + \frac{1}{2}a_0x^4 + \frac{4}{15}a_1x^5 + ... \\ &= a_0[1 + x^2 + \frac{1}{2}x^4 + ...] + a_1[x + \frac{2}{3}x^3 + \frac{4}{15}x^5 + ...]. \end{aligned}$$

Therefore,

$$y_1 = 1 + x^2 + \frac{1}{2}x^4 + ...$$

and

$$y_2 = x + \frac{2}{3}x^3 + \frac{4}{15}x^5 + ...$$

12. Give the definition and an example of: (a) Laplace transform, (b) steady-state solution, (c) ordinary point, (d) forced responce, (e) unstable node, (f) eigenvector.

    Remark: See the book for definitions and examples.

Good luck!