Math 274 Solutions to Practice Exam 2

1. (a) A mass weighing 1 *lb* stretches a spring $\frac{1}{2}$ *ft*. The mass is pulled down 1 *ft* from its equilibrium position and set in motion with an initial downward velocity of 4 *ft/sec*. Suppose there is no air resistance. Find the position $u(t)$ of the mass at any time t, where $u = 0$ corresponds to the equilibrium position where the mass and spring do not move. Also find the amplitude of the spring's motion. Recall that the acceleration due to gravity is 32 *ft/sec*^{2}.

Solution: Since the weight is 1 *lb*, and the acceleration g of gravity is 32 *ft/sec*^{2}, the mass m of the weight satisfies $1 = mg = 32m$, so the mass $m = 1/32$ *lb-sec*^{2}*/ft*. At the equilbrium length $L = 1/2$ *ft*, we have

$$1 = mg = kL = \frac{1}{2}k,$$

where k is the spring constant, so $k = 2$ *lbs/ft*. The general equation of motion of the spring is $mu'' + ku = 0$, hence here $\frac{1}{32}u'' + 2u = 0$, or equivalently,

$$u'' + 64u = 0.$$

This is a linear, constant coefficient, homogeneous equation, so we solve it by looking at the associated characteristic equation $r^2 + 64 = 0$. We get $r^2 = -64$, hence $r = \pm 8i$. Therefore

$$u(t) = C_1 \cos 8t + C_2 \sin 8t.$$

The problem states that the spring is pulled down 1 *ft*, hence $u(0) = 1$. From the formula for $u(t)$, substituting $t = 0$ gives

$$1 = u(0) = C_1 \cos 0 + C_2 \sin 0 = C_1,$$

so $C_1 = 1$. Also the spring is released with a downward velocity of 4 *ft/sec*, so $u'(0) = 4$. We calculate $u'(t) = -8C_1 \sin 8t + 8C_2 \cos 8t$, hence

$$4 = u'(0) = -8C_1 \sin 0 + 8C_2 \cos 0 = 8C_2,$$

so $C_2 = 1/2$. Substituting the values of C_1 and C_2 into the formula for $u(t)$ above, we get

$$u(t) = \cos 8t + \frac{1}{2} \sin 8t,$$

measured in feet. Then the amplitude of the spring's motion, also measured in feet, is

$$A = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{1 + \frac{1}{4}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}.$$

(b) Solve the previous problem using the Laplace transform.

Solution: From the first part we have the equation

$$u'' + 64u = 0.$$

Take the Laplace transform of both sides, denote $\mathcal{L}(u(t))(s) = U(s)$, to obtain

$$s^2\,U(s) - s\,u(0) - u'(0) + 64\,U(s) = 0,$$

or equivalently,

$$s^2\,U(s) - s - 4 + 64\,U(s) = 0.$$

Solving for U, we get $U(s) = \dfrac{s+4}{s^2+64}$. Split it into two fractions $\dfrac{s}{s^2+64}$ and $\dfrac{1}{2}\dfrac{8}{s^2+64}$ and use the inverse Laplace transform (from the table) to get

$$u(t) = \cos 8t + \frac{1}{2}\sin 8t.$$

Check that it's the same as in the first part!

2. (a) A mass of weight 16 *lbs* is attached to a spring with spring constant $k = 8$ *lbs/ft*. If the mass is acted upon by an external force of $14\sin 3t$ *lbs*, and there is no damping, find the differential equation for the position $u(t)$ of the mass at time t, where $u = 0$ corresponds to the equilibrium position. Then find the general solution of this equation.

Solution: Because the weight is 16 *lbs*, the mass satisfies $16 = mg = m \cdot 32$, or $m = \frac{1}{2}$ *lbs-sec*^{2}*/ft*. The general equation of motion for an undamped, forced spring is $mu'' + ku = F(t)$, where F is the forcing function. Here, we obtain

$$\frac{1}{2}u'' + 8u = 14\sin 3t,$$

or equivalently,

$$u'' + 16u = 28\sin 3t.$$

To solve, we first consider the homogeneous equation $u'' + 16u = 0$. The associated characteristic equation is $r^2 + 16 = 0$, or $r^2 = -16$, or $r^2 = \pm 4i$. Therefore the homogeneous solution is $u_h = C_1\cos 4t + C_2\sin 4t$.

To find a particular solution u_p of $u'' + 16u = 28\sin 3t$, we try a solution of the form $u_p = A\cos 3t + B\sin 3t$. These terms are not in common with the terms of u_h, so this form of u_p is sufficient. Differentiating, we get $u_p{}' = -3A\sin 3t + 3B\cos 3t$ and then $u_p{}'' = -9A\cos 3t - 9B\sin 3t$. Substituting into the equation $u'' + 16u = 28\sin 3t$, we get

$$-9A\cos 3t - 9B\sin 3t + 16A\cos 3t + 16B\sin 3t = 28\sin 3t,$$

or $7A\cos 3t + 7B\sin 3t = 28\sin 3t$. Hence $A = 0$ and $B = 4$, so $u_p = 4\sin 3t$.

Finally, $u(t) = u_h(t) + u_p(t) = C_1\cos 4t + C_2\sin 4t + 4\sin 3t$.

(b) Suppose that initially the object was given a speed of 4 *in/sec* and was placed at the equilibrium position. Solve the previous problem using the Laplace transform.

Solution: As in the first problem, take the equation from the above

$$\frac{1}{2}u'' + 8u = 14\sin 3t,$$

and take the Laplace transform of it to obtain

$$\frac{1}{2}\,(s^2\,U(s) - s\,u(0) - u'(0)) + 8\,U(s) = 14\,\frac{3}{s^2+9}.$$

Substitute the initial data (observe that $u(0) = 0$ and $u'(0) = 4/12 ft/sec$)

$$\frac{1}{2}\,(s^2\,U(s) - \frac{1}{3}) + 8\,U(s) = \frac{14\cdot 3}{s^2+9}$$

and simplify

$$s^2\,U(s) + 16\,U(s) = \frac{28\cdot 3}{s^2+9} + \frac{1}{3}.$$

Then solve for $U(s)$ to get

$$U(s) = 28\cdot\frac{3}{(s^2+9)(s^2+16)} + \frac{1}{3}\frac{1}{s^2+16}.$$

Before taking the inverse Laplace transform, we perform partial fraction decomposition to split the first fraction into

$$28\cdot\left(\frac{1}{7}\frac{3}{s^2+9} - \frac{3}{7}\frac{1}{s^2+16}\right) = 4\,\frac{3}{s^2+9} - 3\,\frac{4}{s^2+16}.$$

Hence,

$$U(s) = 4\,\frac{3}{s^2+9} - 3\,\frac{4}{s^2+16} + \frac{1}{3}\frac{1}{s^2+16}.$$

Therefore, the solution is

$$u(t) = 4\sin 3t - 3\sin 4t + \frac{1}{12}\sin 4t = 4\sin 3t - \frac{35}{12}\sin 4t.$$

Observe that this is exactly what you would get if in the first part solve for C_1 and C_2 with the given initial conditions.

Question. Can you identify what is homogeneous solution and what is the particular solution? Also what is the amplitide of the forced response?

3. Transform the equation $u''' - 3u'' + 5u = 0$ into a system of first order equations. If this system is written in the form $\mathbf{x}' = A\mathbf{x}$, what is the matrix A?

   Solution: Let $x_1 = u, x_2 = u'$, and $x_3 = u''$. Then $x_1{}' = u' = x_2$, while $x_2{}' = u'' = x_3$, and

   $$x_3{}' = u''' = 3u'' - 5u = 3x_3 - 5x_1 = -5x_1 + 3x_3,$$

   using the equation $u''' - 3u'' + 5u = 0$. So we have the system $\begin{cases} x_1{}' &= x_2 \\ x_2{}' &= x_3 \\ x_3{}' &= -5x_1 + 3x_3 \end{cases}$ or

   $$\mathbf{x}' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & 0 & 3 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\mathbf{x},$$

   where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & 0 & 3 \end{pmatrix}$.

4. Transform the system $\begin{cases} x_1' = 4x_1 + x_2 \\ x_2' = 3x_1 - 5x_2 \end{cases}$ into a single differential equation of second order.

   Solution: Solving the first equation for x_2, we get $x_2 = x_1{}' - 4x_1$. Taking the derivative (with respect to t) of this equation, we get $x_2{}' = x_1{}'' - 4x_1{}'$. Substituting these values of x_2 and $x_2{}'$ into the equation $x_2{}' = 3x_1 - 5x_2$, we get

   $$x_1{}'' - 4x_1{}' = 3x_1 - 5(x_1{}' - 4x_1) = 3x_1 - 5x_1{}' + 20x_1 = 23x_1 - 5x_1{}',$$

   or, simplifying,

   $$x_1{}'' + x_1{}' - 23x_1 = 0.$$

5. Let A be the matrix $\begin{pmatrix} 3 & 5 \\ 4 & 4 \end{pmatrix}$.

   (a) Find the eigenvalues and associated eigenvectors of A.

   Solution: We have $A - \lambda I = \begin{pmatrix} 3-\lambda & 5 \\ 4 & 4-\lambda \end{pmatrix}$, hence det $(A - \lambda I) = (3-\lambda)(4-\lambda) - 20 = 12 - 7\lambda + \lambda^2 - 20 = \lambda^2 - 7\lambda - 8 = (\lambda - 8)(\lambda + 1) = 0$ when $\lambda = -1$ and $\lambda = 8$. So the eigenvalues of A are -1 and 8.

To find the eigenvector for $\lambda = -1$, we solve $(A-(-1)I)v = 0$, or $\begin{pmatrix} 4 & 5 \\ 4 & 5 \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which gives the equation $4a + 5b = 0$. One choice of non-zero solution is $a = -5$ and $b = 4$, so we get the eigenvector $v = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$.

For $\lambda = 8$, we solve $(A - 8I)v = 0$, or $\begin{pmatrix} -5 & 5 \\ 4 & -4 \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which gives the equation $4a - 4b = 0$, or $a = b$. One choice of non-zero solution is $a = 1$ and $b = 1$, so we get the eigenvector $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) Find two linearly independent solutions to the equation $\mathbf{x}' = A\mathbf{x}$.

<u>Solution:</u> We have $x_1 = e^{-t} \begin{bmatrix} -5 \\ 4 \end{bmatrix}$ and $x_2 = e^{8t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Observe that the determinant of

$$\begin{pmatrix} -5\,e^{-t} & e^{8t} \\ 4\,e^{-t} & e^{8t} \end{pmatrix}$$

is equal to $-9e^{7t}$ which is never equal to zero. So the solutions x_1 and x_2 are indeed linearly independent.

(c) Select one of the linearly independent solutions that you obtained in part B, and check that it actually satisfies the system $\mathbf{x}' = A\mathbf{x}$.

<u>Solution:</u> For example, for $x_1 = \begin{bmatrix} -5e^{-t} \\ 4e^{-t} \end{bmatrix}$, we compute ${x_1}' = \begin{bmatrix} 5e^{-t} \\ -4e^{-t} \end{bmatrix}$, while

$$Ax_1 = \begin{pmatrix} 3 & 5 \\ 4 & 4 \end{pmatrix} \begin{bmatrix} -5e^{-t} \\ 4e^{-t} \end{bmatrix} = \begin{bmatrix} -15e^{-t} + 20e^{-t} \\ -20e^{-t} + 16e^{-t} \end{bmatrix} = \begin{bmatrix} 5e^{-t} \\ -4e^{-t} \end{bmatrix},$$

so ${x_1}'$ agrees with Ax_1.

(d) Write a general solution and sketch the graph of possible trajectories of solutions with orientation.

<u>Solution:</u> Since one of the eigenvectors is positive and one is negative the sketch of trajectories of the solutions to this system will look like a saddle (unstable) with invariant sets being lines of directions $v_1 = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution is

$$u(t) = c_1 \begin{bmatrix} -5 \\ 4 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{8t}.$$

(e) Find the particular solution to $\mathbf{x}' = A\mathbf{x}$ that satisfies $\mathbf{x}(0) = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$.

Solution: Since x_1 and x_2 from part B are linearly independent solutions, the general solution to $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = c_1 e^{-t} \begin{bmatrix} -5 \\ 4 \end{bmatrix} + c_2 e^{8t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\begin{bmatrix} -3 \\ 6 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} -5 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This gives the two equations $-5c_1 + c_2 = -3$ and $4c_1 + c_2 = 6$. If we subtract the second equation from the first we get $-9c_1 = -9$, hence $c_1 = 1$. Then the first equation gives $c_2 = -3 + 5c_1 = -3 + 5 = 2$. Substituting these values of c_1 and c_2, we have $x = e^{-t} \begin{bmatrix} -5 \\ 4 \end{bmatrix} + 2e^{8t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

6. Let $A = \begin{pmatrix} -2 & 6 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 \\ 1 & 4 \end{pmatrix}$.

(a) Find AB.
Solution:

$$AB = \begin{pmatrix} -2 & 6 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -2 \cdot 2 + 6 \cdot 1 & -2 \cdot 2 + 6 \cdot 4 \\ -1 \cdot 2 + 3 \cdot 1 & -1 \cdot 2 + 3 \cdot 4 \end{pmatrix}$$

$$= \begin{pmatrix} -4+6 & -4+24 \\ -2+3 & -2+12 \end{pmatrix} = \begin{pmatrix} 2 & 20 \\ 1 & 10 \end{pmatrix}$$

(b) Find BA.
Solution:

$$BA = \begin{pmatrix} -6 & 18 \\ -6 & 18 \end{pmatrix}.$$

Notice that $AB \neq BA$. Moreover, both AB and BA are singular (why?).

(c) Find A^2.
Solution:

$$A^2 = A \cdot A = \begin{pmatrix} -2 & 6 \\ -1 & 3 \end{pmatrix}.$$

Observe that $A^2 = A$ and $A \neq 0$ nor $A \neq I$!

(d) Is A invertible? Explain how you reach your answer.

Solution: Since $\det A = \begin{vmatrix} -2 & 6 \\ -1 & 3 \end{vmatrix} = -2 \cdot 3 - (-1) \cdot 6 = -6 + 6 = 0$, we see that A is not invertible.

7. Solve the following linear system of algebraic equations: $$\begin{aligned} x_1 - x_2 - 2x_3 &= 1 \\ -x_1 + 2x_2 + 4x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 4 \end{aligned}$$

   Solution: If we add the first and second equations and put the result in place of the second equation, and then add -2 times the first equation to the third equation, and put that result in place of the third equation, we get the equivalent system $\begin{aligned} x_1 - x_2 - 2x_3 &= 1 \\ 0 + x_2 + 2x_3 &= 1 \\ 0 + 3x_2 + 5x_3 &= 2 \end{aligned}$. Now we take -3 times the second of these equations and add it to the third to get the system $\begin{aligned} x_1 - x_2 - 2x_3 &= 1 \\ 0 + x_2 + 2x_3 &= 1 \\ 0 + 0 - x_3 &= -1 \end{aligned}$ From the last equation, we get $x_3 = 1$. Then from the second equation we get $x_2 = 1 - 2x_3 = 1 - 2 \cdot 1 = -1$. From the first equation we obtain $x_1 = 1 + x_2 + 2x_3 = 1 - 1 + 2 = 2$. Hence $x_1 = 2, x_2 = -1$ and $x_3 = 1$.

8. (a) Find two linearly independent real-valued solutions to the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \mathbf{x} .$$

Solution: For $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, we have $A - \lambda I = \begin{pmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix}$. Hence $\det(A - \lambda I) = (1-\lambda)(3-\lambda) - (-2) = 3 - 4\lambda + \lambda^2 + 2 = \lambda^2 - 4\lambda + 5 = 0$ when

$$\lambda = \frac{-(-4) \pm \sqrt{16 - 4 \cdot 5}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

Taking one of the eigenvalues, say $\lambda = 2 + i$, we find the corresponding eigenvector by solving $(A - (2+i)I)v = 0$, or $\begin{pmatrix} -1-i & -1 \\ 2 & 1-i \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This gives the equation $(-1-i)a - b = 0$ (if you multiply this equation by $-1+i$ you get the second equation $2a + (1-i)b = 0$, so it is equivalent), or equivalently $(1+i)a + b = 0$. So we can choose $v = \begin{bmatrix} -1 \\ 1+i \end{bmatrix}$. Therefore a solution is

$$\mathbf{x} = e^{(2+i)t} \begin{bmatrix} -1 \\ 1+i \end{bmatrix} = e^{2t} e^{it} \begin{bmatrix} -1 \\ 1+i \end{bmatrix} = e^{2t}(\cos t + i \sin t) \begin{bmatrix} -1 \\ 1+i \end{bmatrix}$$

$$= e^{2t} \begin{bmatrix} -\cos t \\ \cos t - \sin t \end{bmatrix} + ie^{2t} \begin{bmatrix} -\sin t \\ \cos t + \sin t \end{bmatrix}.$$

The real and imaginary parts of this solution must also be solutions, so we get the two real-valued solutions:

$$x_1 = e^{2t}\begin{bmatrix} -\cos t \\ \cos t - \sin t \end{bmatrix} \text{ and } x_2 = e^{2t}\begin{bmatrix} -\sin t \\ \cos t + \sin t \end{bmatrix}.$$

(b) Show that the two solutions you found in part A are indeed linearly independent.
<u>Solution:</u> We compute the Wronskian of x_1 and x_2:

$$W[x_1, x_2](t) = \begin{vmatrix} -e^{2t}\cos t & -e^{2t}\sin t \\ e^{2t}\cos t - e^{2t}\sin t & e^{2t}\cos t + e^{2t}\sin t \end{vmatrix}$$

$$= -e^{4t}\cos^2 t - e^{4t}\cos t\sin t + e^{4t}\cos t \sin t - e^{4t}\sin^2 t = -e^{4t}(\cos^2 t + \sin^2 t) = -e^{4t}.$$

Since $e^{4t} \neq 0$, the solutions x_1 and x_2 are linearly independent.

9. Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}$. Find the general solution to the system $\mathbf{x}' = A\mathbf{x}$.

<u>Solution:</u> Since we are given that the characteristic polynomial $\det(A-\lambda I)$ is $-(\lambda-2)^2(\lambda-4)$, we see that $\lambda = 2$ is an eigenvalue of multiplicity 2, whereas $\lambda = 4$ is an eigenvalue of multiplicity 1. We have $A - \lambda I = \begin{pmatrix} 3-\lambda & 1 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 3-\lambda \end{pmatrix}$.

To find the eigenvector corresponding to $\lambda = 4$, we solve $(A-4I)v = 0$, or $\begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The second row gives the equation $-2b = 0$, hence $b = 0$. Given this, the first row gives $-a + c = 0$ and the third row gives the equivalent equation $a - c = 0$. So $a = c$, and so one choice of eigenvector is $v = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Hence one solution of $\mathbf{x}' = A\mathbf{x}$ is $x_1 = e^{4t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = 2$, we solve $(a - 2I)v = 0$, or $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This gives only one equation $a + b + c = 0$. Hence we can take b and c to be arbitrary, as long as $a = -b - c$.

So $v = \begin{bmatrix} -b-c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. So $\lambda = 2$ has two linearly independent eigenvectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Hence two solutions of $\mathbf{x}' = A\mathbf{x}$ are $x_2 = e^{2t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $x_3 = e^{2t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Finally, the general solution of $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = c_1x_1 + c_2x_2 + c_3x_3$ or

$$\mathbf{x} = c_1e^{4t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2e^{2t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3e^{2t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

10. For the Laplace transform, practice problems p. 322 #1-10 and #11-23.

    Make sure you practice the inverse Laplace transform and how to complete the square. For example, $x^2 + 8x + 25 = (x+4)^2 + 9$.