SOLUTIONS to Practice Exam 1

1. (a) Find the general solution of $y' = -\frac{3x^2y+x}{x^3+y^3+1}$. You may leave your answer in implicit form.

The equation is the same as $(3x^2y+x)+(x^3+y^3+1)\frac{dy}{dx} = 0$. Let $M = 3x^2y+x$ and $N = x^3+y^3+1$. Then $M_y = 3x^2$ and $N_x = 3x^2$. Hence this equation is exact. Therefore there is a function $\psi(x,y)$ satisfying $\psi_x = M$ and $\psi_y = N$. To find ψ, we first solve

$$\psi_x = M = 3x^2y + x$$

to get

$$\psi = x^3y + \frac{x^2}{2} + h(y).$$

Then all we need is to know h. For this, we compute

$$\psi_y = x^3 + h'(y) = N = x^3 + y^3 + 1.$$

Hence $h'(y) = y^3 + 1$. So $h(y) = \frac{y^4}{4} + y$. So substituting h into the formula for ψ above,

$$\psi = x^3y + \frac{x^2}{2} + \frac{y^4}{4} + y.$$

Hence the solution is $\psi = C$, or

$$x^3y + \frac{x^2}{2} + \frac{y^4}{4} + y = C.$$

(b) Find the particular solution to the equation in part (a) that satisfies $y(0) = 1$. You may leave this answer in implicit form also.

Substituting $x = 0$ and $y = 1$ gives

$$0^3 \cdot 1 + \frac{0^2}{2} + \frac{1^4}{4} + 1 = C,$$

or $\frac{1}{4} + 1 = C$. So $C = \frac{5}{4}$ and the solution is

$$x^3y + \frac{x^2}{2} + \frac{y^4}{4} + y = \frac{5}{4}.$$

2. (a) Find the general solution of

$$\frac{dy}{dx} = \frac{x^5 + 2y}{x}.$$

Your answer should be in explicit form, that is, in the form $y = f(x)$.

The equation can be rewritten as $y' = x^4 + \frac{2}{x}y$ or $y' - \frac{2}{x}y = x^4$. Hence it is a first order linear differential equation. To solve it, we first find the integrating factor

$$\mu(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\int \frac{1}{x}\,dx} = e^{-2\ln|x|} = e^{\ln(|x|^{-2})} = |x|^{-2} = \frac{1}{|x|^2} = \frac{1}{x^2}.$$

Multiplying the equation $y' - \frac{2}{x}y = x^4$ by $\frac{1}{x^2}$ gives

$$\frac{1}{x^2}y' - \frac{2}{x^3}y = x^2,$$

or

$$\frac{d}{dx}\left(\frac{1}{x^2}y\right) = x^2.$$

Hence

$$\frac{1}{x^2}y = \int x^2\,dx = \frac{x^3}{3} + C.$$

Multiplying by x^2 we get

$$y = \frac{x^5}{3} + Cx^2.$$

(b) Check your answer to part (a) by plugging your formula for y into the equation in part (a) and seeing if the equation is satisfied.

Since $y = \frac{1}{3}x^5 + Cx^2$, we have $\frac{dy}{dx} = \frac{5}{3}x^4 + 2Cx$. We also have

$$\frac{x^5+2y}{x} = \frac{x^5 + 2(\frac{1}{3}x^5 + Cx^2)}{x} = \frac{x^5 + \frac{2}{3}x^5 + 2Cx^2}{x} = \frac{\frac{5}{3}x^5 + 2Cx^2}{x} = \frac{5}{3}x^4 + 2Cx.$$

Hence $\frac{dy}{dx} = \frac{x^5+2y}{x}$. That is, the equation in part (a) is satisfied by $y = \frac{1}{3}x^5 + Cx^2$.

3. (a) Find the general solution to the equation $\frac{dy}{dt} - y^5 = 0$, with $y(0) \neq 0$. Your answer should be in explicit form, that is, in the form $y = f(t)$.

We can write this equation in the form $\frac{dy}{dt} = y^5$, and separate variables to obtain $y^{-5}\,dy = dt$. Integrating both sides yields

$$-\frac{1}{4}y^{-4} = t + C_1.$$

Multiplying by -4 and letting $C = -4C_1$, we get

$$\frac{1}{y^4} = y^{-4} = -4t - 4C_1 = -4t + C = C - 4t.$$

Hence

$$y^4 = \frac{1}{C-4t},$$

and finally

$$y = \pm\frac{1}{(C-4t)^{1/4}} \qquad \text{or} \qquad y = \pm\frac{1}{\sqrt[4]{(C-4t)}}.$$

(b) Find the particular solution to the equation in part (a) that satisfies $y(0) = y_0$, where $y_0 \neq 0$ is a given number.

Substituting $x = 0$ and $y = y_0$ into the equation $\frac{1}{y^4} = C - 4t$ gives $\frac{1}{y_0^4} = C$. Hence

$$y = \pm\frac{1}{\sqrt[4]{\frac{1}{y_0^4} - 4t}},$$

where the + sign is taken if $y_0 > 0$ and the negative sign when $y_0 < 0$.

(c) For what values of t is your solution in part (b) defined?

For y to be defined, we need $\frac{1}{y_0^4} - 4t > 0$ (we need ≥ 0 to take the fourth root, and we cannot have 0 because this term is in the denominator). Therefore, the solution is defined for $t < \frac{1}{4y_0^4}$. In other words, the solution is defined on the interval $(-\infty, \frac{1}{4y_0^4})$.

4. Solve the equation $xy' = y - xe^{y/x}$. Your answer should be in explicit form, that is, in the form $y = f(x)$.

Dividing by x, we can write this equation as $y' = \frac{y}{x} - e^{y/x}$. Thus the equation is homogeneous and we make the substitution $v = y/x$. Then $y = xv$, hence by the product rule

$$y' = \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

So after changing variables the equation becomes

$$v + x\frac{dv}{dx} = v - e^v,$$

or

$$x\frac{dv}{dx} = -e^v.$$

The variables can be separated, yielding

$$-\frac{dv}{e^v} = \frac{dx}{x}.$$

Integrating both sides yields (write the integral of the left side as $-\int e^{-v}dv$)

$$e^{-v} = \ln|x| + C.$$

Taking the natural logarithm of both sides gives

$$-v = \ln[C + \ln|x|],$$

hence

$$v = -\ln[C + \ln|x|].$$

Recalling that $v = y/x$, we have

$$\frac{y}{x} = -\ln[C + \ln|x|],$$

hence finally

$$y = -x\ln[C + \ln|x|].$$

5. Solve

$$y'' - 4y' + 5y = 0, \qquad y(0) = 3, \qquad y'(0) = 4.$$

We write the characteristic equation

$$r^2 - 4r + 5 = 0.$$

We cannot factor this, so using the quadratic formula we get

$$r = \frac{4 \pm \sqrt{16 - 4 \cdot 5}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm \sqrt{4}\sqrt{-1}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

This implies that the general solution is

$$y = C_1e^{2t}\cos t + C_2e^{2t}\sin t.$$

Then the condition $y(0) = 3$ yields

$$3 = C_1e^0\cos 0 + C_2e^0\sin 0 = C_1 + 0 = C_1.$$

Next, we calculate y' using the product rule:

$$y' = C_1 2e^{2t}\cos t - C_1 e^{2t}\sin t + C_2 2e^{2t}\sin t + C_2 e^{2t}\cos t.$$

Hence the condition $y'(0) = 4$ yields

$$4 = 2C_1 - 0 + 0 + C_2.$$

Since $C_1 = 3$, this means $C_2 = 4 - 6 = -2$. So finally

$$y = 3e^{2t}\cos t - 2e^{2t}\sin t.$$

6. Consider the equation

$$(\star) \qquad t^2y'' - 5ty' + 8y = 0.$$

(a) Check that $y_1 = t^2$ is a solution of $(\star)$.

Since $y_1 = t^2$, we calculate $y_1' = 2t$ and $y_1'' = 2$. Therefore

$$t^2y_1'' - 5ty_1' + 8y_1 = t^2 \cdot 2 - 5t \cdot 2t + 8 \cdot t^2 = 2t^2 - 10t^2 + 8t^2 = 0.$$

So y_1 is a solution.

(b) Find the general solution of $(\star)$.

To get the general solution, we need two linearly independent solutions. Since we have one solution, we use the method of reduction of order. We guess a second solution of the form $y_2 = vy_1$, where v is a function of t. Here $y_2 = vt^2 = t^2v$; hence by the product rule, $y_2' = 2tv + t^2v'$ and $y_2'' = 2v + 2tv' + 2tv' + t^2v'' = 2v + 4tv' + t^2v''$. Substituting this in the equation, we get

$$\begin{aligned} &t^2y_2'' - 5ty_2' + 8y_2 \\ &= t^2\left[2v + 4tv' + t^2v''\right] - 5t\left[2tv + t^2v'\right] + 8t^2v \\ &= 2t^2v + 4t^3v' + t^4v'' + -10t^2v - 5t^3v' + 8t^2v \\ &= t^4v'' + (4t^3 - 5t^3)v' + (2t^2 - 10t^2 + 8t^2)v \\ &= t^4v'' + -t^3v' = 0. \end{aligned}$$

Dividing by t^4, we get the equation $v'' - \frac{1}{t}v' = 0$. We make the substitution $u = v'$. This gives the equation $u' - \frac{1}{t}u = 0$, which is a first order linear equation which can be solve either by separation of variables or by the method of the integrating factor. For example, we use the second method. The integrating factor is

$$e^{\int -\frac{1}{t}\,dt} = e^{-\ln t} = e^{\ln(t^{-1})} = t^{-1} = \frac{1}{t}.$$

Multiplying through the equation $u' - \frac{1}{t}u = 0$ by the integrating factor $\frac{1}{t}$ gives

$$\frac{1}{t}u' - \frac{1}{t^2}u = 0.$$

The left side is $\left(\frac{1}{t}u\right)'$, by the product rule. So we have $\left(\frac{1}{t}u\right)' = 0$, which implies $\frac{1}{t}u = C$, or $u = Ct$. Then since $u = v'$, we get $v = \int u = \int Ct\,dt = Ct^2/2$. Then

$$y_2 = vt^2 = C\frac{t^2}{2}t^2 = \frac{C}{2}t^4.$$

Since we are just looking for one y_2, we can take $C = 2$ and $y_2 = t^4$. Then the general solution is $y = C_1y_1 + C_2y_2$, or

$$y = C_1t^2 + C_2t^4.$$

7. Find the general solution of
$$y'' + 2y' - 3y = -9t^2 + 8e^t.$$

First we find the solution y_h of the homogeneous equation $y'' + 2y' - 3y = 0$. We guess $y_h = Ce^{rt}$, which leads to the characteristic equation $r^2 + 2r - 3 = 0$. We factor this as $(r + 3)(r - 1) = 0$, so $r = +1$ and -3. So $y_h = C_1e^t + C_2e^{-3t}$.

Now we use the method of undetermined coefficients to try to find a particular solution y_p of the nonhomogeneous problem. Based on the right side $g(t) = -9t^2 + 8e^t$, our first guess is $y_p = At^2 + Bt + C + De^t$. However, the term De^t is in common with the term C_1e^t in y_h, so we have to modify this part of our guess for y_p by multiplying by t. So we get
$$y_p = At^2 + Bt + C + Dte^t.$$
Then
$$y_p' = 2At + B + De^t + Dte^t$$
and
$$y_p'' = 2A + De^t + De^t + Dte^t = 2A + 2De^t + Dte^t.$$
Substituting this in the left side of the inhomogeneous equation, we get
$$\begin{aligned} &y_p'' + 2y_p' - 3y_p \\ &= 2A + 2De^t + Dte^t + 2(2At + B + De^t + Dte^t) - 3(At^2 + Bt + C + Dte^t) \\ &= 2A + 2De^t + Dte^t + 4At + 2B + 2De^t + 2Dte^t - 3At^2 - 3Bt - 3C - 3Dte^t \\ &= -3At^2 + (4A - 3B)t + (2A + 2B - 3C) + 4De^t + (D + 2D - 3D)te^t. \end{aligned}$$
Note that the last term is 0. Setting $-3At^2 + (4A - 3B)t + (2A + 2B - 3C) + 4De^t$ equal to $-9t^2 + 8e^t$ gives the equations
$$-3A = -9 \qquad 4A - 3B = 0 \qquad 2A + 2B - 3C = 0 \qquad 4D = 8.$$
The last gives $D = 2$. The first gives $A = 3$. Substituting $A = 3$ into the second equation gives $3B = 4A = 12$, so $B = 4$. Then the third equation gives $3C = 2A + 2B = 2 \cdot 3 + 2 \cdot 4 = 6 + 8 = 14$, so $C = 14/3$. Hence
$$y_p = At^2 + Bt + C + Dte^t = 3t^2 + 4t + \frac{14}{3} + 2te^t.$$
Therefore the general solution is $y = y_h + y_p$, or
$$y = C_1e^t + C_2e^{-3t} + 3t^2 + 4t + \frac{14}{3} + 2te^t.$$

8. For the equation
$$y'' - 4y' + 4y = t^3e^{2t} + t^2e^{3t} + t\sin 4t,$$
write down the general form of the particular solution y_p that you would use in the method of undetermined coefficients. Just write down the form; do not attempt to solve for the coefficients.

We first find the y_h, the solution to the homogeneous equation $y'' - 4y' + 4y = 0$. For this we guess $y_h = e^{rt}$, which leads to the characteristic equation $r^2 - 4r + 4 = 0$. Factoring, we get $(r - 2)^2 = 0$, so $r = 2$ is a double root of the characteristic equation. Then we know that the general form of y_h is $y_h = c_1e^{2t} + c_2te^{2t}$.

Now we look for a particular solution y_p. Based on the right side $t^3e^{2t} + t^2e^{3t} + t\sin 4t$ of the equation, our first guess for y_p is

$$y_p = (A_1t^3 + A_2t^2 + A_3t + A_4)e^{2t} + (B_1t^2 + B_2t + B_3)e^{3t} + (C_1t + C_2)\sin 4t + (D_1t + D_2)\cos 4t.$$

However, comparing with y_h, we see that the term A_4e^{2t} is in common with the term c_1e^{2t} in y_h, also the term A_3te^{2t} is in common with c_2te^{2t}, so we have to modify the part $(A_1t^3 + A_2t^2 + A_3t + A_4)e^{2t}$ that corresponds to the term t^3e^{2t} on the right side of the equation. Multiplying by t is not sufficient, so we multiply by t^2 to obtain $(A_1t^3 + A_2t^2 + A_3t + A_4)t^2\,e^{2t}$ Thus, we get

$$y_p = (A_1t^3 + A_2t^2 + A_3t^4 + A_4t^5)e^{2t} + (B_1t^2 + B_2t + B_3)e^{3t} + (C_1t + C_2)\sin 4t + (D_1t + D_2)\cos 4t.$$

Another way to look at the modifications of y_p is the following: we multiply by t the first part $(A_1t^3 + A_2t^2 + A_3t + A_4)e^{2t}$, and obtain $t(A_1t^3 + A_2t^2 + A_3t + A_4)e^{2t}$, which still has the term A_4te^{2t} in common with the term C_2te^{2t} in y_h. So we have to multiply by t again to get $t^2(A_1t^3 + A_2t^2 + A_3t + A_4)e^{2t}$. (then there is no repetition in y_p from y_h). The other terms in y_p are unchanged since they correspond to different parts of the right side of the equation. Hence, we obtain

$$y_p = t^2(A_1t^3 + A_2t^2 + A_3t + A_4)e^{2t} + (B_1t^2 + B_2t + B_3)e^{3t} + (C_1t + C_2)\sin 4t + (D_1t + D_2)\cos 4t,$$

which is similar to the above.

9. Assume the following fact: the functions $y_1 = t^2$ and $y_2 = t^{-1}$ are solutions of

$$(\star) \qquad y'' - \frac{2}{t^2}y = 0, \qquad t > 0.$$

(a) Prove that y_1 and y_2 are linearly independent, and write down the general form of the solution of $(\star)$.
Since y_1 and y_2 are solutions of a linear second order ode, to see that they are linearly independent we only have to check that their Wronskian is not 0. We have $y_1 = t^2, y_1' = 2t, y_2 = t^{-1}$, and $y_2' = -t^{-2}$, so

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = t^2 \cdot (-t^{-2}) - t^{-1} \cdot 2t = -1 - 2 = -3,$$

which is not 0. So y_1 and y_2 are linearly independent, hence the general form of the solution is $y_h = C_1y_1 + C_2y_2 = C_1t^2 + C_2t^{-1}$.

(b) Find the general solution of

$$y'' - \frac{2}{t^2}y = 3t^3, \qquad t > 0.$$

To find the general solution, we need to find a particular solution y_p. Using the method of variation of parameters, we guess a solution of the form $y_p = u_1y_1 + u_2y_2$. The equations determining u_1' and u_2' are $u_1'y_1 + u_2'y_2 = 0$ and $u_1'y_1' + u_2'y_2' = g(t)$, where $g(t)$ is the right side of the equation. Using our values of y_1, y_2 and g, we get

$$u_1't^2 + u_2'\frac{1}{t} = 0$$

$$u_1'2t - u_2'\frac{1}{t^2} = 3t^3.$$

We multiply the first equation by $-2t$ to get $-2u_1't^3 - 2u_2' = 0$ and multiply the second equation by t^2 to get $u_1'2t^3 - u_2' = 3t^5$. Then when we add these equations, the u_1' terms cancel to give

$$-3u_2' = 3t^5,$$

or $u_2' = -t^5$. Substituting $u_2' = -t^5$ into $-2u_1't^3 - 2u_2' = 0$ gives $-2u_1't^3 = 2u_2' = -2t^5$ or $u_1' = t^2$. Integrating the equation $u_2' = -t^5$ gives $u_2 = -t^6/6$ while integrating $u_1' = t^2$ gives $u_1 = t^3/3$ (the constants of integration are not needed here). Hence

$$y_p = u_1y_1 + u_2y_2 = \frac{t^3}{3}t^2 - \frac{t^6}{6}\frac{1}{t} = \frac{t^5}{3} - \frac{t^5}{6} = t^5\left(\frac{1}{3} - \frac{1}{6}\right) = \frac{t^5}{6}.$$

Then the general solution is $y = y_h + y_p$, or

$$y = C_1t^2 + C_2t^{-1} + \frac{t^5}{6}.$$

10. Suppose that a certain population obeys the logistic equation patterns:

(a) $\frac{dy}{dt} = -\alpha y(1 - \frac{y}{K})(1 - \frac{y}{T})$ with $T > K$.
(a) Determine equilibrium points of this equation and provide the stability analysis. (b) What happens to the population if $y_0 > T$? if $K < y_0 < T$?

<u>Solution:</u> Solve $-\alpha y(1 - \frac{y}{K})(1 - \frac{y}{T}) = 0$ to obtain critical points $y = 0$, $y = T$ and $y = K$. Graph the arrow (graph is not included here, but you have to draw it on the exam!) to get that $y = 0$ and $y = T$ are stable equilibria and $y = K$ is unstable. Next sketch the picture for y versus t.
If $y_0 > T$, then the population will decrease in size to the level of T. If $K < y_0 < T$, then the population will be increasing to the size of T. Note also that if the initial size is below K, then the population will die out.

(b) $\frac{dy}{dt} = y^2(1 - y)$.
(a) Do the stability analysis; (b) Solve the equation; (c) Find what happens to the population in the long run if $y_0 = .5$.

<u>Solution:</u> Solve $y^2(1 - y) = 0$ to obtain the critical points $y = 0$ and $y = 1$. Observe that $y = 1$ is stable equilibrium and $y = 0$ is semi-stable (stable on the left and unstable on the right).
Next, use separation of variables to get $\frac{dy}{y^2(1-y)} = dt$ and then apply partial fraction decomposition:

$$\frac{Ay + B}{y^2} + \frac{C}{1 - y} = \frac{1}{y^2(1 - y)}.$$

Solve to obtain $Ay - Ay^2 + B - By + Cy^2 = 1$, and thus, $A = 1$, $B = 1$ and $C = 1$. Integrate

$$\int \frac{(y + 1)dy}{y^2} - \int \frac{dy}{y - 1} = \int dt$$

to get

$$\ln|y| - \frac{1}{y} - \ln|y - 1| = t + c \quad \textit{or} \quad \ln\left|\frac{y}{y - 1}\right| - \frac{1}{y} = t + c.$$

The last is the implicit solution.
The population will grow to the size $y = 1$ if the initial size was $y_0 = .5$ (from the graph).