# Math 274 Solutions to Exam 2

1. A.) Are the vectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v_3} = \begin{bmatrix} 1 \\ 8 \\ -4 \end{bmatrix}$ linearly independent or linearly dependent? If they are linearly dependent, find a linear relation among them; that is, find numbers $a, b, c$, not all zero, such that $a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = \mathbf{0}$. If they are not, what can be said about the above numbers $a$, $b$, and $c$ ?

   **Solution:** We determine whether the system $a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = \mathbf{0}$ has a solution other than the trivial solution $a = b = c = 0$. So we try to solve

$$a \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 8 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ or, equivalently } \begin{array}{c} a + 2b + c = 0 \\ -2a + b + 8c = 0 \\ 2a + b - 4c = 0 \end{array}.$$

   If we multiply the first equation by 2, add it to the second equation, and put the result in place of the second equation, and also multiply the first equation by -2, add it to the third equation and put the result in place of the third equation, we get the system

$$\begin{array}{c} a + 2b + c = 0 \\ 0 + 5b + 10c = 0 \\ 0 - 3b - 6c = 0 \end{array}.$$

   The second and third equations are equivalent, and give $b + 2c = 0$, or $b = -2c$. Then the first equation gives $a = -2b - c = 4c - c = 3c$. Taking, for example, $c = 1$, we get a solution with $a = 3, b = -2$, and $c = 1$. So $\mathbf{v_1}, \mathbf{v_2}$, and $\mathbf{v_3}$ satisfy the non-trivial linear relataionship $3\mathbf{v_1} - 2\mathbf{v_2} + \mathbf{v_3} = \mathbf{0}$, and hence they are linearly dependent.

   B.) Is the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 2 & 1 & -4 \end{pmatrix}$ invertible?

   (Notice that the columns of $A$ are the vectors $\mathbf{v_1}, \mathbf{v_2}$, and $\mathbf{v_3}$ from the first part of this problem.)

   **Solution:** The fact shown in part A that the columns are linearly dependent implies that the matrix $A$ is not invertible. Another way to check invertibility is to calculate the determinant, det $= 0$, and thus, the matrix is not invertible.

2. A mass weighing 16 *lb* stretches a spring 4 *ft*. The mass is attached to a viscous damper with damping coefficient 2 *lb-sec/ft*.

   A.) Suppose no external force (except for gravity) acts on the spring. In this part find the differential equation for the position $u(t)$ of the mass at any time $t$, where $u = 0$ corresponds to the equilibrium position where the mass and spring do not move.

   **Solution:** Since the weight is 16 *lb*, and the acceleration $g$ of gravity is 32 $ft/sec^2$, the mass $m$ of the weight satisfies $16 = mg = 32m$, so $m = 16/32 = 1/2$ $lb\text{-}sec^2/ft$. At the equilbrium length $L = 4$ *ft*, we have $16 = mg = kL = 4k$, where $k$ is the spring constant, so $k = 4$ *lbs/ft*. The general equation of motion of the spring is $mu'' + \gamma u' + ku = 0$, where we are given that $\gamma = 2lb - sec/ft$, hence here $\frac{1}{2}u'' + 2u' + 4u = 0$, or equivalently,

$$u'' + 4u' + 8u = 0.$$

B.) Find the general solution to the differential equation you found in part A.

**Solution:** Since $u'' + 4u' + 8u = 0$ is a linear, constant coefficient, homogeneous equation, so we solve it by looking at the associated characteristic equation $r^2 + 4r + 8 = 0$. By the quadratic formula, we get

$$r = \frac{-4 \pm \sqrt{16 - 4 \cdot 8}}{2} = \frac{-4 \pm \sqrt{-16}}{2} = \frac{-4 \pm 4i}{2} = -2 \pm 2i.$$

Therefore, the general solution is

$$u(t) = c_1 e^{-2t} \cos 2t + c_2 e^{-2t} \sin 2t.$$

C.) Now suppose the same spring is acted on by an exernal force of $15\, e^{-t}$ *lbs.* Find the differential equation satisfied by $u(t)$.

**Solution:** In the presence of an external force $F(t)$, the general equation of motion of the spring is $mu'' + \gamma u' + ku = F(t)$, hence, here

$$\frac{1}{2} u'' + 2u' + 4u = 15e^{-t},$$

or

$$u'' + 4u' + 8u = 30e^{-t}.$$

D.) Find the general solution to the differential equation you found in part C.

**Solution:** For a linear, inhomogeneous differential equation, the general solution is $u = u_h + u_p$, where $u_h = c_1 e^{-2t} \cos 2t + c_2 e^{-2t} \sin 2t$ is the homogeneous solution that we found in part C.), and $u_p$ is a particular solution of $u'' + 4u' + 8u = 30e^{-t}$. To find $u_p$, we try a solution of the form $u_p = Ae^{-t}$. This is not in common with the terms of $u_h$, so this form of $u_p$ is sufficient. Differentiating, we get

$$u_p{}' = -Ae^{-t} \quad \text{and} \quad u_p{}'' = Ae^{-t}.$$

Substituting into the equation $u_p{}'' + 4u_p' + 8u_p = 30e^{-t}$, we get

$$Ae^{-t} - 4Ae^{-t} + 8Ae^{-t} = 30e^{-t},$$

or $5A = 30$. Hence, $A = 6$ and so $u_p = 6e^{-t}$. Finally,

$$u(t) = u_h(t) + u_p(t) = c_1 e^{-2t} \cos 2t + c_2 e^{-2t} \sin 2t + 6e^{-t}.$$

E.) Suppose that initially the object had no initial velocity but was stretched down 6 *in.* Find the solution to this initial value problem. (You may use **any** method here.)

**Solution:** From this description we obtain the initial conditions:

$$u(0) = 6\, in = \frac{1}{2}\, ft \quad \text{and} \quad u'(0) = 0.$$

Now, we use the solution from the previous part D.)

$$u(t) = u_h(t) + u_p(t) = c_1 e^{-2t} \cos 2t + c_2 e^{-2t} \sin 2t + 6e^{-t}$$

and find the coefficients $c_1$ and $c_2$:

$$\begin{cases} c_1 + 6 = \frac{1}{2} \\ -2\,c_1 + 2\,c_2 - 6 = 0 \end{cases}$$

Thus, $c_1 = -\frac{11}{2}$ and $c_2 = -\frac{5}{2}$ and the solution is

$$u(t) = -\frac{11}{2}\, e^{-2t} \cos 2t + -\frac{5}{2}\, e^{-2t} \sin 2t + 6e^{-t}.$$

3. (15 points) Find the general solution of the system and draw a sketch of solutions trajectories.

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & -3 \\ 1 & 2 & -1 \end{pmatrix} \mathbf{x}.$$

**Solution:** Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & -3 \\ 1 & 2 & -1 \end{pmatrix}$. Then $A - \lambda I = \begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 4-\lambda & -3 \\ 1 & 2 & -1-\lambda \end{pmatrix}$. Hence,

$$\det(A - \lambda I) = (2-\lambda)[(4-\lambda)(-1-\lambda)+6] = (2-\lambda)[-4+\lambda-4\lambda+\lambda^2+6)$$
$$= (2-\lambda)(\lambda^2-3\lambda+2) = (2-\lambda)(\lambda-2)(\lambda-1) = -(\lambda-2)^2(\lambda-1).$$

Hence $\lambda = 2$ is an eigenvalue of multiplicity 2 and $\lambda = 1$ is an eigenvalue of multiplicity 1.

To find the eigenvector corresponding to $\lambda = 1$, we solve $(A - 1 \cdot I)v = 0$, or

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & -3 \\ 1 & -2 & -2 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The first row gives the equation $a = 0$. Given this, the second row gives $3b - 3c = 0$ and the third row gives the equivalent equation $b - c = 0$. So $b = c$, and so one choice of eigenvector is $v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Thus, one solution of $\mathbf{x}' = A\mathbf{x}$ is $x_1 = e^t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = 2$, we solve $(a - 2I)v = 0$, or $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & 2 & -3 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This gives only one equation $a + 2b - 3c = 0$. Hence, we can take $b$ and $c$ to be arbitrary, as long as $a = -2b + 3c$. So $v = \begin{bmatrix} -2b+3c \\ b \\ c \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. So $\lambda = 2$ has two linearly independent eigenvectors $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. Thus, two solutions of $\mathbf{x}' = A\mathbf{x}$ are $x_2 = e^{2t} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $x_3 = e^{2t} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

Finally, the general solution of $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = c_1 x_1 + c_2 x_2 + c_3 x_3$, or

$$\mathbf{x} = c_1 e^t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Observe that all eigenvalues are positive, therefore, all solutions are unstable and expanding. So any plot with expanding trajectories from the origin will work here.

4. Find functions $x_1$ and $x_2$ satisfying $\begin{cases} x_1' = -2x_1 + x_2 \\ x_2' = 7x_1 + 4x_2 \end{cases}$ and the initial conditions $x_1(0) = 5, x_2(0) = 3$.

**Solution:** In matrix form, the equation is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{pmatrix} -2 & 1 \\ 7 & 4 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{with} \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

Letting $A = \begin{pmatrix} -2 & 1 \\ 7 & 4 \end{pmatrix}$, we have $A - \lambda I = \begin{pmatrix} -2-\lambda & 1 \\ 7 & 4-\lambda \end{pmatrix}$. Hence, det $(A - \lambda I) = (-2 - \lambda)(4-\lambda) - 7 = -8 - 4\lambda + 2\lambda + \lambda^2 - 7 = \lambda^2 - 2\lambda - 15 = (\lambda - 5)(\lambda + 3) = 0$ when $\lambda = 5$ and $\lambda = -3$. So the eigenvalues of $A$ are

$$\lambda_1 = 5 \quad \text{and} \quad \lambda_2 = -3.$$

To find the eigenvector for $\lambda_1 = 5$, we solve $(A - 5I)v = 0$, or

$$\begin{pmatrix} -7 & 1 \\ 7 & -1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which gives the equation $-7x + y = 0$. Thus, $v_1 = \begin{bmatrix} x \\ 7x \end{bmatrix}$. One choice of non-zero solution is $x = 1$ and $y = 7$, so we get the eigenvector $v_1 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$. This gives one solution

$$x_1 = e^{5t} \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

to the system $\mathbf{x}' = A\mathbf{x}$.

For $\lambda_2 = -3$, we solve $(A - (-3)I)v = 0$, or $\begin{pmatrix} 1 & 1 \\ 7 & 7 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which gives the equation $x + y = 0$, or $y = -x$. Thus, $v_2 = \begin{bmatrix} x \\ -x \end{bmatrix}$. One choice of non-zero solution is $x = 1$ and $x = -1$, so we get the eigenvector $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This gives another solution

$$x_2 = e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

to the system $\mathbf{x}' = A\mathbf{x}$.

Thus, the general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = c_1 x_1 + c_2 x_2 = c_1 e^{5t} \begin{bmatrix} 1 \\ 7 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

From the initial condition $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, we get

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ 7c_1 - c_2 \end{bmatrix}.$$

This gives the equations $c_1 + c_2 = 5$ and $7c_1 - c_2 = 3$. Adding these equations gives $8c_1 = 8$, hence $c_1 = 1$. Then from $c_1 + c_2 = 5$, we get $c_2 = 4$. Substituting these in the formula for $\mathbf{x}$, we obtain

$$\mathbf{x} = e^{5t} \begin{bmatrix} 1 \\ 7 \end{bmatrix} + 4e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

or $x_1 = e^{5t} + 4e^{-3t}$ and $x_2 = 7e^{5t} - 4e^{-3t}$.

5. A.) Find two linearly independent **real-valued** solutions to the system

$$\mathbf{x}' = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \mathbf{x} \, .$$

**Solution:** For $A = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix}$, we have $A - \lambda I = \begin{pmatrix} 3-\lambda & 4 \\ -2 & -1-\lambda \end{pmatrix}$. Hence, $\det(A - \lambda I) = (3-\lambda)(-1-\lambda) - (-2)(4) = -3 + \lambda - 3\lambda + \lambda^2 + 8 = \lambda^2 - 2\lambda + 5 = 0$ when

$$\lambda = \frac{-(-2) \pm \sqrt{4 - 4 \cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Taking one of the eigenvalues, say $\lambda = 1 + 2i$, we find the corresponding eigenvector by solving $(A - (1+2i)I)v = 0$, or $\begin{pmatrix} 2-2i & 4 \\ -2 & -2-2i \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The second component gives the equation $-2a - (2+2i)b = 0$, or equivalently $a + (1+i)b = 0$ (if you multiply this equation by $2 - 2i$ you get the equation $(2-2i)a + 4b = 0$ corresponding to the first component). So we can choose a solution $a = 1+i, b = -1$; that is, $v = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$. Therefore, a solution of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = e^{(1+2i)t} \begin{bmatrix} 1+i \\ -1 \end{bmatrix} = e^t e^{2it} \begin{bmatrix} 1+i \\ -1 \end{bmatrix} = e^t(\cos 2t + i \sin 2t) \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$$

$$= e^t \begin{bmatrix} \cos 2t + i\cos 2t + i \sin 2t - \sin 2t \\ -\cos 2t - i \sin 2t \end{bmatrix} = e^t \begin{bmatrix} \cos 2t - \sin 2t \\ -\cos 2t \end{bmatrix} + ie^t \begin{bmatrix} \cos 2t + \sin 2t \\ -\sin t \end{bmatrix}.$$

The real and imaginary parts of this solution must also be solutions, so we get the two real-valued solutions:

$$x_1 = e^t \begin{bmatrix} \cos 2t - \sin 2t \\ -\cos t \end{bmatrix} \quad \text{and} \quad x_2 = e^t \begin{bmatrix} \cos 2t + \sin 2t \\ -\sin 2t \end{bmatrix}.$$

B.) Show that the two solutions you found in part A are indeed linearly independent.

**Solution:** We compute the Wronskian of $x_1$ and $x_2$:

$$W[x_1, x_2](t) = \begin{vmatrix} e^t(\cos 2t - \sin 2t) & e^t(\cos 2t + \sin 2t) \\ -e^t \cos t & -e^t \sin t \end{vmatrix}$$

$$= -e^{2t}(\cos 2t \sin 2t - \sin^2 2t) + e^{2t}(\cos^2 2t + \sin 2t \cos 2t) = e^{2t}(\sin^2 2t + \cos^2 2t) = e^{2t}.$$

Since $e^{2t} \neq 0$, the solutions $x_1$ and $x_2$ are linearly independent.

6. (TRUE-FALSE)

(1) If the real part of an eigenvalue is negative, then the solutions on the invariant set generated by the corresponding eigenvector are unstable.

FALSE. The solutions are stable, since all trajectories are directed towards the origin ($\lambda < 0$).

(2) A steady-state solution is the homogeneous solution for the second order differential equation with the force.

FALSE. It is a particular solution.

(3) If eigenvalues are complex, then it is possible to have closed ellipses as trajectories of solutions.

TRUE. This can be the case when $Re\lambda = 0$.

(4) The Laplace transform is an integral transform with the kernel $e^{st}$.

FALSE. The kernel is $e^{-st}$.

7. Using the Laplace Transform, solve

$$y'' + 8\,y' + 20\,y = \sin 2t, \quad y(0) = 0, \;\; y'(0) = 5$$

**Solution:** Using the table we transform the equation into the form

$$s^2\,Y(s) - s\,y(0) - y'(0) + 8\,(sY(s) - y(0)) + 20\,Y(s) = \frac{2}{s^2+4},$$

or

$$s^2\,Y(s) - 5 + 8\,sY(s) + 20\,Y(s) = \frac{2}{s^2+4},$$

$$(s^2 + 8s + 20)\,Y(s) = \frac{2}{s^2+4} + 5,$$

and thus,

$$Y(s) = \frac{2}{(s^2+4)(s^2+8s+20)} + \frac{5}{s^2+8s+20}.$$

Observe that $s^2+8s+20$ is "irreducible" i.e. we can not factor and will need to complete square: $s^2+8s+20 = (s+4)^2+4$. Use partial fraction decomposition to split the first fraction into

$$\frac{As+B}{s^2+4} \quad \text{and} \quad \frac{Cs+D}{s^2+8s+20}.$$

After obtaining common denominator, we get the following expression in the numerator $(As+B)(s^2+8s+20)+(Cs+D)(s^2+4)$. Expanding it out and collecting similar coefficients we obtain

$$\begin{cases} s^3: & A + C & = 0 \\ s^2: & 8A + B + D & = 0 \\ s: & 20A + 8B + 4C & = 0 \\ c: & 20B + 4D & = 2 \end{cases}$$

First equation gives $C = -A$, the fourth equation gives $D = \frac{1}{2} - 5B$, substituting this into the second equation and solving for $B$, we obtain $8A - \frac{1}{2} = 4B$, or $B = 2A + \frac{1}{8}$. Substituting all previous expressions into the third equation, we obtain $20A + 16A + 1 - 4A = 0$, solving it, we get $A = -\frac{1}{32}$. Also we obtain $B = \frac{1}{16}$, $C = \frac{1}{32}$ and $D = \frac{3}{16}$. Thus, the solution $Y(s)$ splits as

$$\begin{aligned} Y(s) &= -\frac{1}{32}\cdot\frac{s-2}{s^2+4} + \frac{\frac{1}{32}s + \frac{3}{16}}{(s+4)^2+4} + \frac{5}{(s+4)^2+4} \\ &= \frac{1}{32}\frac{s+2}{s^2+4} + \frac{1}{32}\frac{s+4+2}{(s+4)^2+4} + \frac{5}{(s+4)^2+4} \\ &= \frac{1}{32}\left[\frac{s}{s^2+4} + \frac{2}{s^2+4}\right] + \frac{1}{32}\frac{s+4}{(s+4)^2+4} + \frac{81}{32}\frac{2}{(s+4)^2+4} \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$y(t) = \frac{1}{32}(\cos 2t + \sin 2t + e^{-4t}\cos 2t + 81\,e^{-4t}\sin 2t).$$

8. (BONUS) Transform the equation $u'''' + 2u''' + 7u'' - 3u' - 10u = 0$ with initial condition $u(0) = -4, u'(0) = 9, u''(0) = -5, u'''(0) = 12$ into an equivalent system of first order differential equations. If this system is written in the form $\mathbf{x}' = A\mathbf{x}$, what is the matrix $A$, is it invertible, what is the determinant and what is $\mathbf{x}(0)$?

<u>**Solution:**</u> Let $x_1 = u, x_2 = u', x_3 = u''$, and $x_4 = u'''$. Then $x_1{}' = u' = x_2$, while $x_2{}' = u'' = x_3$, and $x_3{}' = u''' = x_4$, whereas

$$x_4{}' = u'''' = 10u + 3u' - 7u'' - 2u''' = 10x_1 + 3x_2 - 7x_3 - 2x_4,$$

using the equation $u'''' + 2u''' + 7u'' - 3u' - 10u = 0$. So we have the system

$$\begin{aligned} x_1{}' &= x_2 \\ x_2{}' &= x_3 \\ x_3{}' &= x_4 \\ x_4{}' &= 10x_1 + 3x_2 - 7x_3 - 2x_4 \end{aligned} \qquad \text{or } \mathbf{x}' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10 & 3 & -7 & -2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = A\mathbf{x},$$

where $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10 & 3 & -7 & -2 \end{pmatrix}$. Also we have $x_1(0) = u(0) = -4, x_2(0) = u'(0) = 9, x_3(0) = u''(0) = -5$, and $x_4(0) = u'''(0) = 12$, hence $\mathbf{x}(0) = \begin{bmatrix} -4 \\ 9 \\ -5 \\ 12 \end{bmatrix}$. The determinant of this matrix is -10. Therefore, the matrix is invertible.