

Math 274 Exam 1 Solutions

1. (13 points) Find the general solution of the equation $x \frac{dy}{dx} = 2\sqrt{xy} + y$. Your answer should be in explicit form, that is, in the form $y = f(x)$.

Solution: Dividing the equation through by x , we can write it as

$$y' = 2\frac{\sqrt{xy}}{x} + \frac{y}{x} = 2\sqrt{\frac{xy}{x^2}} + \frac{y}{x} = 2\sqrt{\frac{y}{x}} + \frac{y}{x}.$$

Thus, the equation is homogeneous and we make the substitution $v = y/x$. Then $y = xv$, hence by the product rule

$$y' = \frac{dy}{dx} = v + x \frac{dv}{dx}.$$

So after making this substitution, the equation becomes

$$v + x \frac{dv}{dx} = 2\sqrt{v} + v,$$

or, after subtracting v from both sides,

$$x \frac{dv}{dx} = 2\sqrt{v}.$$

Separating variables, we get

$$\frac{dv}{\sqrt{v}} = \frac{2dx}{x}.$$

Integrating both sides yields (using $\int \frac{1}{\sqrt{v}} dv = \int v^{-1/2} dv = 2v^{1/2} = 2\sqrt{v}$), we have

$$2\sqrt{v} = 2 \ln |x| + C_1.$$

Dividing both sides by 2 gives

$$\sqrt{v} = C + \ln |x|,$$

where we have let $C = C_1/2$. Hence

$$v = (C + \ln |x|)^2.$$

Recalling that $v = y/x$, we have

$$\frac{y}{x} = (C + \ln |x|)^2,$$

hence, finally

$$y = x(C + \ln |x|)^2.$$

2. (a) (10 points) Find the general solution of

$$2\frac{dy}{dt} = -e^t y^3.$$

Your answer should be in explicit form, that is, in the form $y = f(t)$.

Solution: Dividing the equation through by y^3 and multiplying by dt , we can write the equation as

$$\frac{2}{y^3} dy = -e^t dt,$$

which shows that the equation is separable. Integrating both sides (using $\int \frac{1}{y^3} dy = \int y^{-3} dy = y^{-2}/(-2) = -1/(2y^2)$) gives

$$\frac{-1}{y^2} = -e^t + C_1.$$

Multiplying the equation through by -1 and letting $C = -C_1$ gives

$$\frac{1}{y^2} = e^t + C.$$

Hence, we get $y^2 = 1/(e^t + C)$ or

$$y = \pm \frac{1}{\sqrt{e^t + C}}.$$

Remark: the plus sign is taken if $y(0) > 0$, the minus sign is taken if $y(0) < 0$, and the solution is just $y = 0$ if $y(0) = 0$, but you are not required to note this.

- (b) (3 points) Find the particular solution to the equation in part (a) that also satisfies $y(0) = 2$.

Solution: Substituting $t = 0$ and $y = 2$ into $y = \frac{1}{\sqrt{e^t + C}}$ (since $y(0) > 0$ we take the plus sign) gives

$$2 = \frac{1}{\sqrt{1 + C}}.$$

Squaring both sides gives

$$4 = \frac{1}{1 + C}.$$

Multiplying by $1 + C$ and dividing by 4 gives $1 + C = 1/4$, hence $C = -3/4$. Therefore, the particular solution is

$$y = \frac{1}{\sqrt{e^t - 3/4}}.$$

- (c) (3 points) Find the interval of existence of the solution you found in part b.

Solution: The function y in part b is defined if and only if $e^t - 3/4 > 0$ (we can't have $e^t - 3/4 < 0$ since we can't take a square root of a negative number, and we can't have $e^t - 3/4 = 0$ since we can't divide by 0). Equivalently, the solution is defined if $e^t > 3/4$. Taking the natural logarithm of both sides (which preserves the inequality because the logarithm function is increasing), we get $t > \ln(3/4)$. So the interval of existence is $I = (\ln(3/4), +\infty)$.

3. (10 points) Find the general solution of $(2x^3y + 3y^2)y' = -3x^2y^2 - 1$. You may leave your answer in implicit form.

Solution: The equation is the same as $(3x^2y^2 + 1) + (2x^3y + 3y^2)y' = 0$. Let $M = 3x^2y^2 + 1$ and $N = 2x^3y + 3y^2$. Then $M_y = 6x^2y$ and $N_x = 6x^2y$. Since $M_y = N_x$, this equation is exact. Therefore, there is a function $\Psi(x, y)$ satisfying $\Psi_x = \frac{\partial F}{\partial x} = M$ and $\Psi_y = \frac{\partial F}{\partial y} = N$. To find Ψ , we first solve

$$\Psi_x = M = 3x^2y^2 + 1 \quad \text{to get} \quad \Psi(x, y) = x^3y^2 + x + h(y).$$

Then all we need to know is $h(y)$. For this, we compute

$$\Psi_y = 2x^3y + h'(y) = N = 2x^3y + 3y^2.$$

Hence, $h'(y) = 3y^2$. Thus, $h(y) = y^3$ (constant is not necessary here, since there will be a constant on the right-hand side). Substituting $h(y)$ into the formula for $\Psi(x, y)$ above, we obtain

$$\Psi(x, y) = x^3y^2 + x + y^3.$$

Hence, the solution is $\Psi(x, y) = C$, or

$$x^3y^2 + x + y^3 = C.$$

4. (9 points) [TRUE-FALSE]

(1) The population equation $\frac{dy}{dt} = y \ln(K/y)$ has 2 equilibria: stable and unstable.

FALSE. There are 2 equilibria $y = 0$ and $y = K$. Note that the equation is defined for $(y > 0)$. The first one is semi-unstable and the second is stable.

(2) The equation $\frac{dy}{dt} = ky$ is the logistic equation in population dynamics.

FALSE. This is not the “logistic” equation, it is the “exponential growth” equation.

(3) The Wronskian $W(g, f)$ has to be non-zero at least at one point in order for f and g to be linearly independent.

TRUE. It suffices to have one point on which the Wronskian is non-zero on the interval where f and g are defined and differentiable.

5. (7 points) For the equation

$$y'' - 2y' + y = t^3 + te^t + t \sin t,$$

write down the general form of the particular solution y_p that you would use in the method of undetermined coefficients. Just write down the form; do not attempt to solve for the coefficients.

Solution: We first find y_h , the solution to the homogeneous equation $y'' - 2y' + y = 0$. The characteristic equation is $r^2 - 2r + 1 = 0$. Factoring, we get $(r - 1)^2 = 0$, so $r = 1$ is a **double** root of the characteristic equation. Thus, the general form of y_h is $y_h = C_1 e^t + C_2 t e^t$.

Now we look for a particular solution y_p . The right side of the equation is of the form $g_1 + g_2 + g_3$ where $g_1(t) = t^3$, $g_2(t) = te^t$, and $g_3(t) = t \sin t$. Corresponding to $g_1 = t^3$ we guess $y_{p_1} = At^3 + Bt^2 + Ct + D$, the general polynomial of degree 3. Since none of these terms is a constant multiple of any term in y_h , this form of y_{p_1} is sufficient. Corresponding to $g_2 = te^t$, we set $y_{p_2} = Ete^t + Fe^t$. However, both terms Ete^{2t} and Fe^{2t} are the same as in y_h , so we need to modify y_{p_2} by taking higher powers of t . Multiplying by t is not sufficient (we still have one repeated term with y_h), but by t^2 does it: we obtain $y_{p_2} = Et^3 e^{2t} + Ft^2 e^{2t}$. This no longer has any overlap with y_h , so this form of y_{p_2} is sufficient. For $g_3 = t \sin t$, we guess $y_{p_3} = (Gt + H) \sin t + (It + J) \cos t$. None of these terms, after you multiply them out, is a constant multiple of any term in y_h , so this form for y_{p_3} is sufficient. Thus, $y_p = y_{p_1} + y_{p_2} + y_{p_3}$ or

$$y_p = At^3 + Bt^2 + Ct + D + (Et^3 + Ft^2)e^{2t} + (Gt + H) \sin t + (It + J) \cos t.$$

6. (10 points) If the Wronskian W of f and g is $t^2 \cos t$ and $f(t) = t$, find $g(t)$.

Solution: We set the Wronskian $W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} t & g \\ 1 & g' \end{vmatrix} = t^2 \cos t$ (we used $f = t$, hence $f' = 1$). Thus, we have

$$tg' - g = t^2 \cos t.$$

This is a first order linear equation which we should solve for g . Dividing the equation through by t , we get

$$g' - \frac{1}{t}g = t \cos t.$$

Then *the integrating factor* is

$$e^{\int p dt} = e^{-\int \frac{dt}{t}} = e^{-\ln |t|} = e^{\ln(|t|^{-1})} = |t|^{-1} = \frac{1}{|t|}.$$

Multiplying the equation $g' - \frac{1}{t}g = t \cos t$ through by $\frac{1}{t}$ gives

$$\frac{1}{t}g' - \frac{1}{t^2}g = \cos t.$$

The left side is $(\frac{1}{t}g)'$, by the product rule, so we have the following equation

$$\left(\frac{1}{t}g\right)' = \cos t.$$

Therefore,

$$\frac{1}{t}g = \int \cos t dt = \sin t + C, \quad \text{or} \quad g(t) = t(\sin t + C) = t \sin t + Ct$$

Note that $+C$ is important here, since it is later on multiplied by t .

7. (a) (4 points) Check that the functions $y_1 = t$ and $y_2 = t^3$ are solutions of

$$t^2 y'' - 3ty' + 3y = 0, \quad t > 0.$$

Solution: Since $y_1 = t$, we get $y_1' = 1$ and $y_1'' = 0$. Therefore, $t^2 \times 0 - 3t \times 1 + 3t = 0 - 3t + 3t = 0$, hence, y_1 is a solution.

For $y_2 = t^3$, we get $y_2' = 3t^2$ and $y_2'' = 6t$. Hence, $t^2(6t) - 3t(3t^2) + 3t^3 = 6t^3 - 9t^3 + 3t^3 = 0$, so y_2 is a solution.

- (b) (15 points) Find the general solution of

$$t^2 y'' - 3ty' + 3y = t^4, \quad t > 0.$$

Solution: First rewrite the equation in the standard form:

$$y'' - \frac{3}{t}y' + \frac{3}{t^2}y = t^2.$$

Note that $g(t) = t^2$.

To find the general solution, we need to find a particular solution y_p . Using the method of variation of parameters, we write $y_p = u_1 y_1 + u_2 y_2$. The equations determining u_1' and u_2'

$$\text{are } \begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(t). \end{cases}$$

Using our values of y_1, y_2 from part (a) and $g(t)$, we get

$$u_1' t + u_2' t^3 = 0$$

$$u_1' \cdot 1 + u_2' 3t^2 = t^2.$$

To solve these equations simultaneously for u_1' and u_2' , it is easiest to first cancel a factor of t from the first equation to get $u_1'^{\text{prime}} + u_2'^{\text{prime}} t^2 = 0$. Then $u_1' = -u_2' t^2$ and substitute this into the second equation: $-u_2' t^2 + u_2' 3t^2 = t^2$ to get $2t^2 u_2' = t^2$. Hence, $u_2' = \frac{1}{2}$, and so, $u_2 = \frac{1}{2}t$ (+ c is not necessary here, since we just need one possible solution u_2).

Substituting $u_2' = 1/2$ into $u_1' = -u_2' t^2$ gives $u_1' = -\frac{1}{2}t^2$. Integrating the equation $u_1' = -\frac{1}{2}t^2$ gives $u_1 = -\frac{1}{2} \frac{t^3}{3} = -\frac{1}{6}t^3$. Therefore,

$$y_p = u_1 y_1 + u_2 y_2 = -\frac{1}{6}t^3 \cdot t + \frac{1}{2}t \cdot t^3 = \left(-\frac{1}{6} + \frac{1}{2}\right)t^4 = \left(-\frac{1}{6} + \frac{3}{6}\right)t^4 = \frac{2}{6}t^4 = \frac{1}{3}t^4.$$

Then the general solution is $y = y_h + y_p$, or

$$y = C_1 t + C_2 t^3 + \frac{1}{3}t^4.$$

8. (16 points) Consider the epidemiological model of Herpes infection given by

$$\frac{dI}{dt} = \beta I \left(1 - \frac{I}{N}\right) - \mu I,$$

where N is the total population, I is the number of people infected by the infection and β is the transmission rate of the disease. Since the disease is not fatal, μ is just the natural mortality rate for the population considered (for example, $\mu \ll \beta$).

- (a) Find any equilibria that exist and discuss the stability.
 (b) Suppose that initially there was only one person infected by Herpes, according to this model how does the infection evolve in a long run?
 (c) Now suppose that initially everyone was infected (bacteriological attack by Herpes!), what happens to the population eventually?

Solution: (I explained this problem in details in class with all the graphs, here there is a brief outline of the solution.)

- (a) Write $\beta I \left(1 - \frac{I}{N}\right) - \mu I = 0$, factor I out and solve for I to obtain:

$$I = 0 \quad \text{and} \quad I = N\left(1 - \frac{\mu}{\beta}\right).$$

Since $\mu \ll \beta$, it means that $1 - \frac{\mu}{\beta} > 0$, and hence, the second critical point $I = N\left(1 - \frac{\mu}{\beta}\right) > 0$ (also notice that it is less than the total population N). Next, draw the graph to obtain the arrows to determine that $I = 0$ is unstable and $I = N\left(1 - \frac{\mu}{\beta}\right)$ is stable.

- (b) The number of infected people increases and eventually stabilizes at the size a little bit less than the entire population $N\left(1 - \frac{\mu}{\beta}\right)$.
 (c) The number of infected will decrease with time, dropping down to the equilibrium, i.e. to the size of $N\left(1 - \frac{\mu}{\beta}\right)$ people.

9. (10 points) Find the general solution of

$$y^{(4)} - 4y''' + 13y'' = 0.$$

Solution: The characteristic equation is $r^4 - 4r^3 + 13r^2 = 0$ which simplifies to $r^2(r^2 - 4r + 13) = 0$. Equating both factors to zero, we get $r^2 = 0$ (thus, $r = 0$ is a **double** root) and $r^2 - 4r + 13 = 0$, which is solved by the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{c} = \frac{4 \pm \sqrt{16 - 4 \times 13}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

Hence, the general solution is

$$y_h = c_1 + c_2 t + c_3 e^{2t} \cos(3t) + c_4 e^{2t} \sin(3t).$$