Lagrange Multipliers

This method is an alternative way of calculating extrema (min/max) points of a surface $f(x,y)$ that is constrained to a path defined by $g(x,y)=c$. The underlying mathematics that 'makes this work' involves a development of vectors, which we won't go into here. However, the result is a neat little algorithm that allows for such calculations without having to do direct substitution, which often results in brutal and potentially impossible algebra.

So here is the Method of Lagrange Multipliers, in a handy wallet-size version:

Start: You'll be given a function $z = f(x,y)$ and a constraint $g(x,y)=c$.

> **Note:** The constraint can be viewed as a contour on the *xy*-plane, and when it is projected onto the surface $z = f(x,y)$, it appears as a path on the actual surface. <u>Our goal is to find how high or low this path gets.</u>

Step 1: Form the Lagrange function. The form is:

$$L(x,y) = f(x,y) - \lambda(g(x,y) - c)$$

Distribute the lambda (λ) through to remove the parentheses.

Step 2: Find partial derivatives L_x and L_y. Set these equal to zero. You will have two equations in three variables: *x*, *y* and λ.

Step 3 (very important): Eliminate the lambda! Isolate the λ in both equations, then set the two results equal to one another. This leaves a relation/equation in terms of the variables *x* and *y*. Lambda λ is gone. It's job is finished. Reduce your equation as much as possible.

Step 4: Take your result in step 3 and combine (by substitution) with the given constraint $g(x,y)=c$. You should now be able to algebraically reduce this to a single variable, and thus, solve for that variable. Work backwards to find the other variables and the resulting z-values.

Example: Find the extrema points on $f(x,y) = x^2 + xy + 2y^2 + x + 2y + 1$ when constrained to the path given by $2x + y = 1$.

Step 1: Form the Lagrange function: $L(x,y) = x^2 + xy + 2y^2 + x + 2y + 1 - \lambda(2x + y - 1)$. After simplification, we get: $L(x,y) = x^2 + xy + 2y^2 + x + 2y + 1 - 2\lambda x - \lambda y + \lambda$.

Step 2: Find the first partials of *L*:

$$L_x = 2x + y + 1 - 2\lambda$$
$$L_y = x + 4y + 2 - \lambda$$

Step 3: Set the partials equal to zero. Don't skip this step!

$$2x+y+1-2\lambda=0$$
$$x+4y+2-\lambda=0$$

Multiply the bottom equation by –2 and we get:

$$2x+y+1-2\lambda=0$$
$$-2x-8y-4+2\lambda=0$$

Add, and the lambda terms drop out:

$$\begin{matrix}2x+y+1-2\lambda=0\\ -2x-8y-4+2\lambda=0\end{matrix}\Rightarrow -7y-3=0$$

(We got lucky in that the *x*-terms also dropped out, too!)

So therefore, $y=-\frac{3}{7}$.

Step 4: Plug this *y* back into the original constraint to get *x*:

$$2x+y=1\Rightarrow 2x-\frac{3}{7}=1\Rightarrow x=\frac{5}{7}.$$

When we get just a single point, there's no way to tell conclusively if it's a min or a max, but we can recognize that the given surface is a paraboloid that opens up, so any straight line path on it will have a low point, so this point is a min.

Our answer: $\left(\frac{5}{7},-\frac{3}{7},\frac{50}{49}\right)$, min.

Example 2: $f(x,y)=x^2+y^2+x-y+1$, constraint: $x^2+y^2=1$.

Step 1:
$$L(x,y)=x^2+y^2+x-y+1-\lambda x^2-\lambda y^2+\lambda$$

Step 2:
$$L_x=2x+1-2\lambda x$$
$$L_y=2y-1-2\lambda y$$

Step 3:
$$\begin{matrix}2x+1-2\lambda x=0\\ 2y-1-2\lambda y=0\end{matrix}\Rightarrow\begin{matrix}\lambda=\frac{2x+1}{2x}\\ \lambda=\frac{2y-1}{2y}\end{matrix}\Rightarrow\frac{2x+1}{2x}=\frac{2y-1}{2y}$$

(Isolate the λ for both equations, then set the two equal to one another.)

Now, cross multiply, distribute and simplify:

$$\frac{2x+1}{2x}=\frac{2y-1}{2y}\Rightarrow (2x+1)2y=(2y-1)2x\Rightarrow 4xy+2y=4xy-2x\Rightarrow y=-x.$$

Notice that the '4*xy*' terms canceled and the two's then canceled, leaving us with the relation $y=-x$.

Step 4: Plug $y=-x$ into the constraint, and solve for *x*:

$$x^2+(-x)^2=1\Rightarrow 2x^2=1\Rightarrow x=\pm\frac{\sqrt{2}}{2}$$

We now have two *x* values. We plug them into the constraint $x^2+y^2=1$, one at a time:

When $x=\frac{\sqrt{2}}{2}$, we get $\left(\frac{\sqrt{2}}{2}\right)^2+y^2=1$, which gives $\frac{1}{2}+y^2=1$ and finally, $y^2=\frac{1}{2}$. Taking the square root we get two possible *y* values: $y=\pm\frac{\sqrt{2}}{2}$. We now have two points:

$$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},2\right)\quad \&\quad \left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},2+\sqrt{2}\right)$$

We do the exact same thing for $x=-\frac{\sqrt{2}}{2}$. This gives us two more *y*-values: $y=\pm\frac{\sqrt{2}}{2}$, and we generate two more points:

$$\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},2-\sqrt{2}\right)\quad \&\quad \left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},2\right)$$

Comparing *z*-values we conclude that our max occurs at $\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},2+\sqrt{2}\right)$ and our min occurs at $\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},2-\sqrt{2}\right)$.

See graph, next page.

Math 211 – Lagrange Multipliers Method