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Diagonally implicit multistage integration methods for pseudospectral solutions of the wave equation

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Abstract

Diagonally implicit multistage integration methods are employed for the numerical integration in time of first order hyperbolic systems arising from Chebyshev pseudospectral discretizations of the spatial derivatives in the wave equation. These methods have stage order q equal to the order p . The stage values can be utilized to recover approximations to the solution u of sufficiently high accuracy. The phenomenon of order reduction, which is present in the integration of differential systems by numerical methods of low stage order, such as explicit Runge–Kutta methods, is avoided. © 2000 IMACS. Published by Elsevier Science B.V. All rights reserved.

Keywords: Pseudospectral Chebyshev method; Hyperbolic systems; Wave equation; Eigenvalue stability; General linear methods; Order and stage order

1. Introduction

In [15] we proposed a variant of a pseudospectral method for the numerical solution of the one- or two-dimensional wave equation on a physically unbounded domain. To illustrate the approach we consider in [15] the use of low order absorbing boundary conditions to provide a numerical solution on an artificially bounded domain. The wave equation is first reformulated as a first order hyperbolic system with the boundary conditions incorporated appropriately. Spatial derivatives are discretized using the pseudospectral Chebyshev method, and the resulting system is integrated in time by the classical Runge–Kutta (RK) method of order four. Eigenvalue stability analysis confirmed, by numerical experiments, that this approach is about 30% more efficient than the formulation adopted in [21] which is based on the solution of the wave equation as a second order hyperbolic equation. The disadvantage of the approach proposed in [15] is the difficulty of recovering the solution u from the computed approximations to u_t to a sufficiently high order of accuracy. The classical RK method of order four has stage order one (see, for example, [1,13]). Thus the computed approximations to the stage values are only low order approximations to u_t computed at the grid points. As a consequence, to recover u with a sufficient

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accuracy, appropriate approximations to u can only be recovered at every third step. Moreover, the resulting multistep formula for an approximation to u is only weakly stable which can lead to a further loss of accuracy.

It is the purpose of this paper to circumvent the above difficulties by integrating the differential systems resulting from spatial discretization by diagonally implicit multistage integration methods (DIMSIMs), a subclass of general linear methods for ordinary differential equations (ODEs). These methods were first proposed by Butcher in [2] and further investigated by Butcher et al. [3,8], Butcher and Jackiewicz [4–7], Jackiewicz and Vermiglio [16], Jackiewicz et al. [17,18], and Van Wieren [24–26]. For these methods the computed stage values can be used to recover u from u_t at every step of integration, because the stage order q is equal to the order p of the method.

The additional advantage of DIMSIMs over RK methods is the avoidance of the order reduction phenomenon, i.e., the effective reduction of order that occurs when integrating stiff differential systems by methods of low stage order. Theoretically, there is no reduction of order that occurs when integrating differential systems by DIMSIMs, and the observed order of convergence is equal to $p = q = 4$. This is also confirmed by numerical experiments.

The organization of this paper is as follows. In the next section the DIMSIM method of order four is introduced and the details of its implementation are discussed. In Section 3 these methods are applied to the differential systems which arise in the pseudospectral Chebyshev discretization of the spatial derivatives occurring in the discretization of the hyperbolic systems of the first order. In Section 4 results of some numerical experiments for the wave equation in one and two space dimensions are presented. In Section 5 concluding remarks are given, and plans for future research are outlined.

2. Implementation details for DIMSIM of order four

Spatial discretization of partial differential equations of evolution leads to the system of ODEs

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T], \\ y(t_0) = y_0 \in \mathbb{R}^m, \end{cases} \quad (2.1)$$

where the dimension m depends on the number of grid points, which in turn depends on the desired accuracy. Such systems of equations can be integrated in time by many standard methods, including RK (multi-stage) methods, and multi-step methods, see, for example, [1,13]. Here multi-step, multi-stage methods of a particular structure, DIMSIMs, are considered. These are methods of the form

$$\begin{cases} Y_i^{[n]} = h \sum_{j=1}^s a_{ij} f(t_{n-1} + c_j h, Y_j^{[n]}) + \sum_{j=1}^r u_{ij} y_j^{[n-1]}, & i = 1, 2, \dots, s, \\ y_i^{[n]} = h \sum_{j=1}^s b_{ij} f(t_{n-1} + c_j h, Y_j^{[n]}) + \sum_{j=1}^r v_{ij} y_j^{[n-1]}, & i = 1, 2, \dots, r, \end{cases} \quad (2.2)$$

$n = 1, 2, \dots, N$, $Nh = T - t_0$, $t_n = t_0 + nh$. Here $Y_i^{[n]}$ approximate $y(t_{n-1} + c_i h)$, $i = 1, 2, \dots, s$, up to the (stage) order q , and $y_i^{[n]}$ approximate linear combinations

$$\sum_{k=0}^r \alpha_{ik} h^k y^{(k)}(t_n), \quad i = 1, 2, \dots, r, \quad (2.3)$$

with coefficients α_{ik} , up to the order p . As mentioned in Section 1 these methods were introduced in [2] and further investigated in [3–8,16–18,24–26].

To illustrate and compare the utilization of DIMSIMs for systems of type (2.1) the DIMSIM of order four constructed in [5] is considered. For this DIMSIM $p = q = r = s = 4$, $U = [u_{ij}]_{i,j=1}^4 = [\delta_{ij}]_{i,j=1}^4$ is the identity matrix of dimension four, V is a rank one matrix of the form $V = [v \ v \ v \ v]^T$ such that $v^T e = 1$, $e = [1 \ 1 \ 1 \ 1]^T$, and the vector c is chosen as

$$c = [c_1 \ c_2 \ c_3 \ c_4]^T = \left[\frac{1}{8} \ \frac{3}{8} \ \frac{5}{8} \ \frac{7}{8}\right]^T.$$

It was demonstrated in [2] that the coefficient matrix $B = [b_{ij}]_{i,j=1}^4$ is uniquely determined by the relation

$$B = B_0 - AB_1 - VB_2 + VA,$$

where B_0 , B_1 , and B_2 are some matrices which depend only on the vector c (see [2,5] for explicit formulae). In [5] the vector $v = [v_j]_{j=1}^4$, and the coefficient matrix $A = [a_{ij}]_{i,j=1}^4$, are chosen in such a way that the stability function

$$p(w, z) = \det(wI - V - zB(I - zA)^{-1}U)$$

of the resulting method takes the form

$$p(w, z) = w^3(w - R(z)).$$

Here,

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4$$

is the stability polynomial of the 4-stage explicit RK method of order four. This leads to

$$v = [1.063426258 \ -3.090699405 \ 2.269665404 \ 0.7576077337]^T,$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1.087521532 & 0 & 0 & 0 \\ 2.130622781 & 0.1740733143 & 0 & 0 \\ 3.167186705 & -0.08499907798 & 0.3505683223 & 0 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 3.903672365 & 0.639427501 & -0.630929947 & 0.627835727 \\ 4.418277238 & -1.512741959 & 2.842115909 & -2.045167066 \\ 4.480588403 & -2.160830002 & 4.717606355 & -4.302055204 \\ 4.330295982 & -2.093542755 & 5.729033231 & -6.108536762 \end{bmatrix}.$$

Define the matrix W by

$$W = [\alpha_0 \ \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4] = \begin{bmatrix} \alpha_{10} & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{30} & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{40} & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix},$$

where α_{ik} are the coefficients appearing in (2.3). It follows from the order conditions derived in [2] (compare also [6]) that

$$\alpha_0 = e, \quad \alpha_i = \frac{c^i}{i!} - \frac{Ac^{i-1}}{(i-1)!}, \quad i = 1, 2, 3, 4.$$

Here, c^i denotes componentwise multiplication. For the DIMSIM with coefficients c , A , B , U , and V defined above, this leads to

$$W = \begin{bmatrix} 1 & 0.125 & 0.0078125 & 0.00032552083 & 0.000010172526 \\ 1 & -0.7125215338 & -0.06562769172 & 0.01180508371 & 0.0004699636934 \\ 1 & -1.6796960990 & -0.1362928411 & 0.01180508372 & 0.004134325428 \\ 1 & -2.5577559553 & -0.2003163864 & 0.02441612178 & 0.009875650412 \end{bmatrix}.$$

This matrix can be used to compute the components of the starting vector from the formulas

$$y_i^{[0]} = \sum_{k=0}^4 \alpha_{ik} h^k \tilde{y}^{(k)}(t_0), \quad i = 1, 2, 3, 4, \tag{2.4}$$

given the approximations $h^k \tilde{y}^{(k)}(t_0)$ to the scaled derivatives $h^k y^{(k)}(t_0)$ for $k = 0, 1, 2, 3$ and 4.

The approximations to the derivatives up to order 4 at time 0 are calculated as follows: Given a starting stepsize h_s (for example, $h_s = h/4$), the approximations, $y_h(t_0 + jh_s)$, to $y(t_0 + jh_s)$ for $j = 0, 1, 2, 3$ and 4, can be computed by the classical RK method of order four, for which the coefficients are given by Butcher tableau

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & 0 & \frac{1}{2} & \\ \hline 1 & 0 & 0 & 1 \\ \hline \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}, \tag{2.5}$$

compare [1,19]. Expanding $y(t_0 + jh_s) = y_h(t_0 + jh_s) + O(h^5)$ as a Taylor series around t_0 , and dropping the terms of order higher than $O(h^5)$, the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 1 & 2 & 2 & \frac{4}{3} & \frac{2}{3} \\ 1 & 3 & \frac{9}{2} & \frac{9}{2} & \frac{27}{8} \\ 1 & 4 & 8 & \frac{32}{3} & \frac{32}{3} \end{bmatrix} \begin{bmatrix} \tilde{y}(t_0) \\ h_s \tilde{y}'(t_0) \\ h_s^2 \tilde{y}''(t_0) \\ h_s^3 \tilde{y}'''(t_0) \\ h_s^4 \tilde{y}^{(4)}(t_0) \end{bmatrix} = \begin{bmatrix} y_h(t_0) \\ y_h(t_0 + h_s) \\ y_h(t_0 + 2h_s) \\ y_h(t_0 + 3h_s) \\ y_h(t_0 + 4h_s) \end{bmatrix}, \tag{2.6}$$

is obtained. Rescaling $h_s^k \tilde{y}^{(k)}(t_0)$, with $\xi = h/h_s$, gives

$$h_s^k \tilde{y}^{(k)}(t_0) = \frac{1}{\xi^k} h^k \tilde{y}^{(k)}(t_0). \tag{2.7}$$

Substituting (2.7) into (2.6) and solving the resulting system for $h^k \tilde{y}^{(k)}(t_0)$ leads to

$$\begin{bmatrix} \tilde{y}(t_0) \\ h \tilde{y}'(t_0) \\ h^2 \tilde{y}''(t_0) \\ h^3 \tilde{y}'''(t_0) \\ h^4 \tilde{y}^{(4)}(t_0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{25}{12}\xi & 4\xi & -3\xi & \frac{4}{3}\xi & -\frac{1}{4}\xi \\ \frac{35}{12}\xi^2 & -\frac{26}{3}\xi^2 & \frac{57}{6}\xi^2 & -\frac{14}{3}\xi^2 & \frac{11}{12}\xi^2 \\ -\frac{5}{2}\xi^3 & 9\xi^3 & -12\xi^3 & 7\xi^3 & -\frac{3}{2}\xi^3 \\ \xi^4 & -4\xi^4 & 6\xi^4 & -4\xi^4 & \xi^4 \end{bmatrix} \begin{bmatrix} y_h(t_0) \\ y_h(t_0 + h_s) \\ y_h(t_0 + 2h_s) \\ y_h(t_0 + 3h_s) \\ y_h(t_0 + 4h_s) \end{bmatrix}.$$

The values for $h^k \tilde{y}^{(k)}(t_0)$ can then be utilized in (2.4) to compute the components $y_i^{[0]}$ of the starting vector $y^{[0]}$. The approximations $y^{[n]}$, $n = 1, 2, \dots, N$, can then be generated recursively using (2.2). Approximations $\tilde{y}(t_n)$ to $y(t_n)$, at the gridpoints t_n , are computed using the formula

$$\tilde{y}(t_n) = h \sum_{j=1}^4 \beta_{0j} f(t_{n-1} + c_j h, Y_j^{[n]}) + \sum_{j=1}^4 \gamma_{0j} y_j^{[n-1]},$$

which is a special case of the formula

$$h^k \tilde{y}^{(k)}(t_n) = h \sum_{j=1}^4 \beta_{kj} f(t_{n-1} + c_j h, Y_j^{[n]}) + \sum_{j=1}^4 \gamma_{kj} y_j^{[n-1]},$$

$k = 0, 1, 2, 3, 4$, developed in [2,6]. As explained in [6] the matrices $\tilde{B} = [\beta_{kj}]_{k=0, j=1}^4$ and $\tilde{V} = [\gamma_{kj}]_{k=0, j=1}^4$ can be generated from the relation

$$[1 \ z \ z^2 \ z^3 \ z^4]^T \exp(z) = z \tilde{B} \exp(cz) + \tilde{V} w(z) + O(z^5),$$

where

$$\exp(cz) = [\exp(c_1 z) \ \exp(c_2 z) \ \exp(c_3 z) \ \exp(c_4 z)]^T,$$

and $w(z) = \sum_{j=0}^4 \alpha_j z^j$. For our purposes only the first rows $\beta_0 = [\beta_{0j}]_{j=1}^4$ and $\gamma_0 = [\gamma_{0j}]_{j=1}^4$, of the matrices \tilde{B} and \tilde{V} , are needed. It was proved in [24, Theorem 4, p. 25] that the vector β_0 may be chosen to be equal to the first row v^T of the matrix V and γ_0 satisfies the linear system

$$\exp(z) - v^T w(z) = z \gamma_0 \exp(cz) + O(z^5),$$

whose solution is

$$\gamma_0 = [3.981146330 \ 0.323672289 \ -0.138091399 \ 0.248278434].$$

3. The pseudospectral Chebyshev discretization of spatial derivatives

Consider the system (2.1) which arises from reformulation of the two-dimensional wave equation,

$$\begin{cases} u_{tt} = c^2(u_{xx} + u_{yy}), & 0 < x, y < 1, \ t > 0, \\ u(x, y, 0) = f(x, y), & 0 < x, y < 1, \\ u_t(x, y, 0) = 0, & 0 < x, y < 1, \end{cases} \tag{3.1}$$

where c is the dimensionless propagation speed, as follows. Suppose numerical solutions simulating unbounded domains are required. To illustrate the approach, the artificial boundaries at $x = 0, 1$ and $y = 0, 1$ are approximated by the lowest order absorbing boundary conditions

$$\begin{cases} u_t - cu_x = 0, & x = 0, \\ u_t + cu_x = 0, & x = 1, \\ u_t - cu_y = 0, & y = 0, \\ u_t + cu_y = 0, & y = 1 \end{cases} \tag{3.2}$$

(see, for example, [11,14]). Problem (3.1) in conjunction with equations (3.2) is reformulated as a first order hyperbolic system by introducing the notation $U_1 = u_t$, $U_2 = u_x$, and $U_3 = u_y$. Then (3.1) and (3.2), respectively, become

$$\begin{cases} U_{1,t} = c^2(U_{2,x} + U_{3,y}), & 0 < x, y < 1, t > 0, \\ U_{2,t} = U_{1,x}, & 0 < x, y < 1, \\ U_{3,t} = U_{1,y}, & 0 < x, y < 1, \\ U_1(x, y, 0) = 0, & 0 < x, y < 1, \\ U_2(x, y, 0) = f_x(x, y), & 0 < x, y < 1, \\ U_3(x, y, 0) = f_y(x, y), & 0 < x, y < 1, \end{cases} \quad (3.3)$$

and

$$\begin{cases} U_1 - cU_2 = 0, & x = 0, \\ U_1 + cU_2 = 0, & x = 1, \\ U_1 - cU_3 = 0, & y = 0, \\ U_1 + cU_3 = 0, & y = 1. \end{cases} \quad (3.4)$$

The first order spatial derivatives occurring in these equations can be discretized using the pseudospectral Chebyshev method, see, for example, [9,12], based on the Chebyshev–Gauss–Lobatto points

$$x_i = \frac{1}{2} \left(1 - \cos \frac{\pi(i-1)}{n} \right),$$

$i = 1, 2, \dots, n+1$. Then, for example,

$$U_{1,t}(x_i, y_j, t) = c^2 \left(\sum_{k=1}^{n+1} d_{ik} U_2(x_k, y_j, t) + \sum_{k=1}^{n+1} d_{jk} U_3(x_i, y_k, t) \right), \quad t > 0, \quad (3.5)$$

where d_{ij} , $i, j = 1, 2, \dots, n+1$, are elements of the pseudospectral differentiation matrix of order one. This matrix can be efficiently computed by the algorithm described in [12], or by the explicit expression found in [9].

The incorporation of the absorbing boundary conditions (3.4) can be done in many different ways and is subtle. It was demonstrated in [15] that the stability of the overall system depends on the adopted approach. Consequently, the efficiency of the resulting algorithm is also dependent on the adopted approach, via the eigenvalue spectrum of the system matrix (compare [22]). It was demonstrated in [15] that the most efficient approach arises from the incorporation of the absorbing boundary conditions (3.4) into the equations for $U_{2,t}$ and $U_{3,t}$, yielding the system

$$\tilde{U}_t = Q\tilde{U}, \quad t > 0. \quad (3.6)$$

Here,

$$\begin{aligned} \tilde{U}_i(t) &= [U_i(x_1, y_1, t), \dots, U_i(x_{n+1}, y_1, t), \dots, U_i(x_1, y_{n+1}, t), \dots, U_i(x_{n+1}, y_{n+1}, t)]^T, \\ & i = 1, 2, 3, \\ \tilde{U} &= [\tilde{U}_1^T, \tilde{U}_2^T, \tilde{U}_3^T]^T. \end{aligned}$$

The structure of the matrices Q for the one- and two-dimensional problems, and the corresponding initial conditions, are discussed in [15].

Denote by $\tilde{U}^h(t)$ the approximation to the solution $\tilde{U}(t)$ of (3.6). Then

$$\tilde{U}^h(t_{n-1} + c_v h) = \tilde{U}(t_{n-1} + c_v h) + O(h^4), \quad v = 1, 2, 3, 4,$$

because (2.2) with coefficients given in Section 2 has stage order $q = 4$. These approximations to stage values can be used to recover the approximation $\tilde{u}^h(t_n)$ to $\tilde{u}(t_n)$ at every step of integration. Here, $\tilde{u}(t)$ is defined as

$$\tilde{u}(t) = [u(x_1, y_1, t), \dots, u(x_{n+1}, y_1, t), \dots, u(x_1, y_{n+1}, t), \dots, u(x_{n+1}, y_{n+1}, t)]^T,$$

where u is the solution to (3.1). This is in contrast to the situation one encounters while integrating (3.6) by the classical RK method (2.5), which has stage order equal to $q = 1$ only, as was done in [15].

The resulting formula for $\tilde{u}^h(t_n)$ is

$$\begin{aligned} \tilde{u}^h(t_n) = & \tilde{u}^h(t_{n-1}) + h(\beta_1 \tilde{U}_1^h(t_{n-1} + c_1 h) + \beta_2 \tilde{U}_1^h(t_{n-1} + c_2 h) \\ & + \beta_3 \tilde{U}_1^h(t_{n-1} + c_3 h) + \beta_4 \tilde{U}_1^h(t_{n-1} + c_4 h)), \end{aligned}$$

where

$$\beta_1 = \frac{13}{48}, \quad \beta_2 = \frac{11}{48}, \quad \beta_3 = \frac{11}{48}, \quad \beta_4 = \frac{13}{48}.$$

These values of β_i were computed by the method of undetermined coefficients.

4. Numerical examples

Many numerical experiments to evaluate the effectiveness of the DIMSIMs (2.2) described in Section 2 were performed. Fig. 1 illustrates some results of numerical experiments for the one-dimensional wave equation with the initial condition defined by

$$u(x, 0) = f(x) = \exp(-a(x - x_0)) \sin(k(x - x_0)), \tag{4.1}$$

$a = 100$, $x_0 = 0.5$, and $k = 40$. This problem was solved on the time interval $[0, 0.6]$ with $c = 1$, $n + 1 = 64$, and $h = 0.004$, which, as demonstrated in [15], is close to the maximal stepsize which guarantees stable integration. The true solution is plotted by a solid line and numerical approximations are denoted by the symbol ‘o’.

Fig. 2 illustrates some results of numerical experiments for the two-dimensional wave equation with the initial condition

$$f(x, y) = \exp(-a((x - x_0)^2 + (y - y_0)^2)), \tag{4.2}$$

centered at the point (x_0, y_0) . This problem was solved on the time interval $[0, 0.8]$ for $a = 100$, $(x_0, y_0) = (0.75, 0.25)$, $c = 1$, $n + 1 = 32$, and $h = 0.004$, which is again the maximal stepsize which guarantees stable integration for the above value of n . The results of numerical simulations were then rendered on the uniform grid consisting of 64 points in each space dimension as described in [15, Appendix].

This figure looks similar to Fig. 7 in [15]. However the computed numerical approximations \tilde{u}^h to the solution u of (3.1) are more accurate than the approximations obtained in [15]. This is due to the fact that the method (2.2) has stage order q equal to the order $p = 4$ and, as a consequence, does not suffer from the order reduction phenomenon as numerical solution computed by classical RK method does.

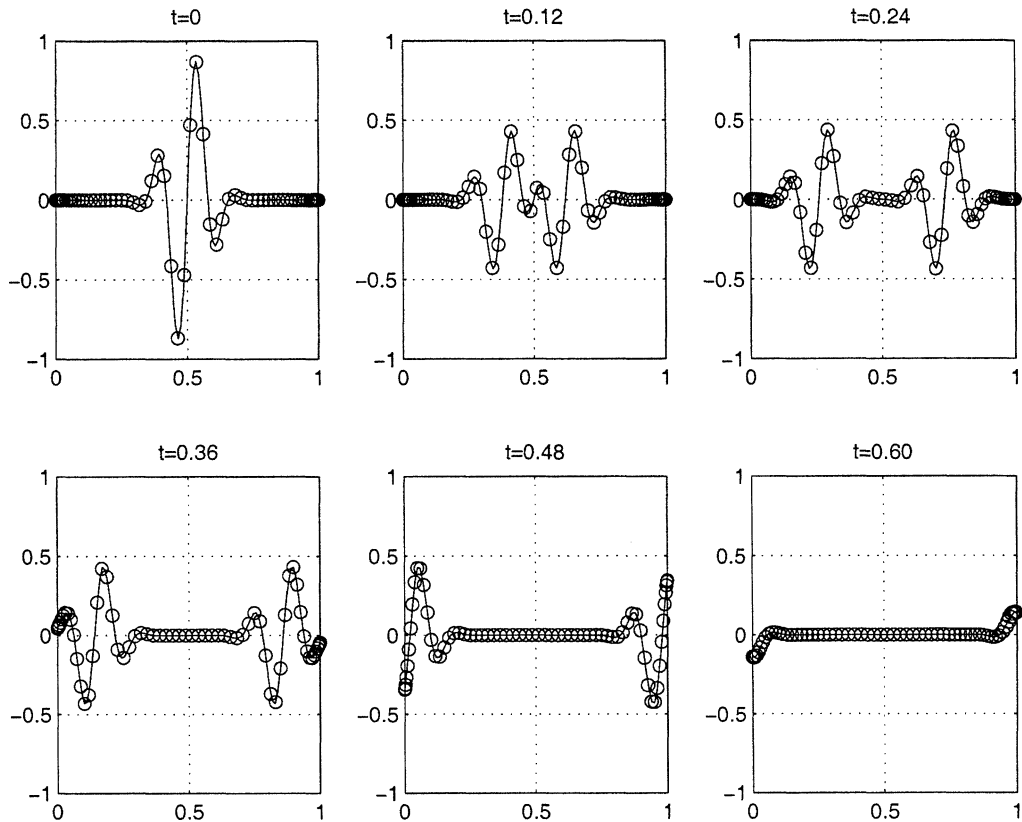


Fig. 1. Exact and numerical solution for the one-dimensional wave equation with initial condition (4.1).

5. Concluding remarks

It has been demonstrated that DIMSIMs (2.2) defined in Section 2 can be effectively used for the numerical solution of the wave equation. These methods have many attractive features which are not possessed by the conventional schemes for integrating systems of ordinary differential equations, such as, for example, RK formulae. They have stage order q equal to the order p and this makes it possible to utilize the approximations to the stage values at the nongrid points $t_n + c_v h$ to recover the approximations of order $p = q$ to the solution $u(t_n)$ at every step of integration. Hence, they do not suffer from the order reduction phenomenon which affects methods of low stage order such as, for example, explicit RK methods. In addition, the methods (2.2) have the advantage that they allow for the construction of continuous interpolants and efficient error estimates of the local discretization error with only minimal additional overhead costs [6]. They can be also efficiently implemented on nonuniform grids using the Nordsieck representation described in [3].

Not all of these features of DIMSIMs were utilized in this paper. The context, however, in which DIMSIMs are discussed here, namely the constant stepsize implementation of a particular DIMSIM, in conjunction with the semi-discretization of the wave equation by the Chebyshev pseudospectral method, is novel. Other related works have considered only the design of efficient RK methods for

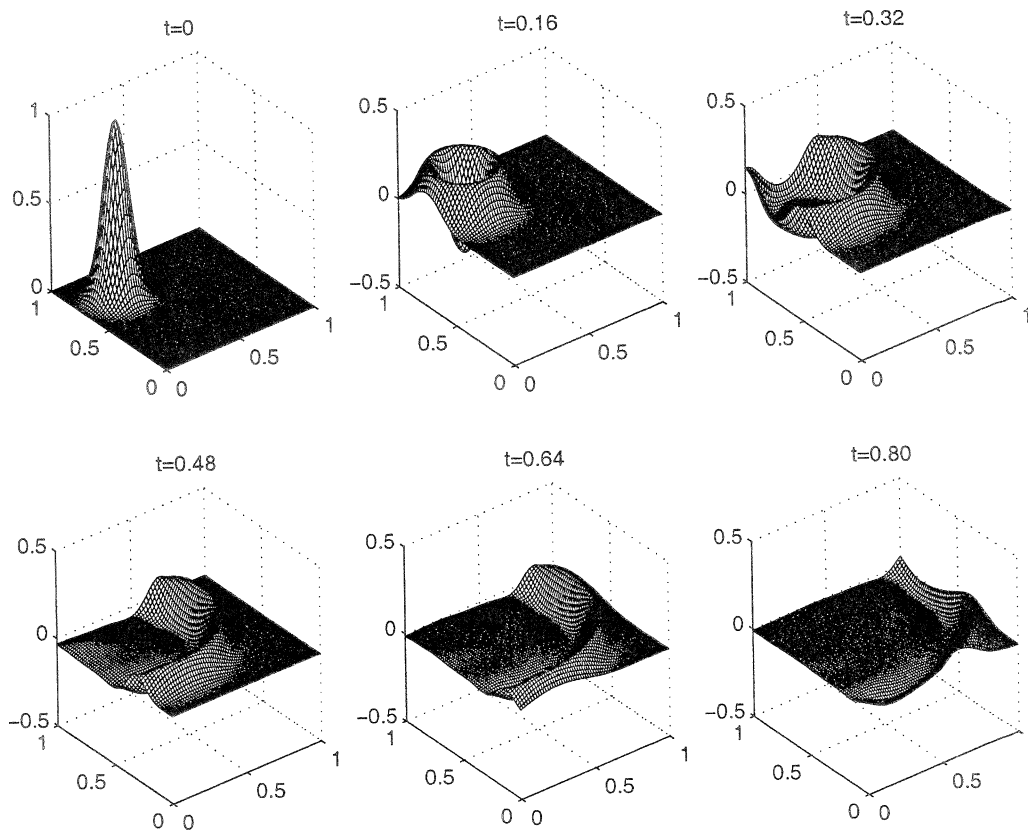


Fig. 2. Numerical results for the two-dimensional wave equation with initial condition (4.2).

integrating either steady state hyperbolic systems, [10] or optimal RK methods minimizing dispersion and/or dissipation [20,23]. Such approaches are not immediately applicable for variable stepsize/variable order implementations that control error. The effectiveness of the DIMSIM solver demonstrated here suggests that incorporation of DIMSIMs in a variable stepsize/variable order implementation is viable. Future work will address the implementation of DIMSIMs for these systems with variable step/variable order environments, and order ranging from one to eight. DIMSIMs of order up to $p = q = 4$ were constructed in [5] using symbolic manipulation software (MATHEMATICA or MAPLE), and numerical algorithms based on a homotopy approach (PITCON, ALCON or HOMEPACK). Methods of order $p = q = 5$ and $p = q = 6$ were found recently in [7] using the approach based on Fourier series, and the use of minimization packages from the MINPACK based on the Levenberg–Marquardt algorithm. DIMSIMs of order $p = q = 7$ and $p = q = 8$ were constructed in [8] using state-of-the-art optimization methods based on variable-model trust-region least-squares algorithms. Various implementation issues for the methods (2.2) are discussed in [3,6,24–26].

It is expected that efficient implementation of DIMSIMs in a variable step/variable order environment will lead to further computational gains as compared with standard RK methods, and the method of order $p = q = 4$ utilized in this paper.

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References

- [1] J.C. Butcher, *The Numerical Analysis of Ordinary Differential Equations. Runge–Kutta and General Linear Methods*, Wiley, Chichester/New York, 1987.
- [2] J.C. Butcher, Diagonally-implicit multi-stage integration methods, *Appl. Numer. Math.* 11 (1993) 347–363.
- [3] J.C. Butcher, P. Chartier, Z. Jackiewicz, Nordsieck representation of DIMSIMs, *Numer. Algorithms* 16 (1997) 209–230.
- [4] J.C. Butcher, Z. Jackiewicz, Diagonally implicit general linear methods for ordinary differential equations, *BIT* 33 (1993) 452–472.
- [5] J.C. Butcher, Z. Jackiewicz, Construction of diagonally implicit general linear methods of type 1 and 2 for ordinary differential equations, *Appl. Numer. Math.* 21 (1996) 385–415.
- [6] J.C. Butcher, Z. Jackiewicz, Implementation of diagonally implicit multistage integration methods for ordinary differential equations, *SIAM J. Numer. Anal.* 34 (1997) 2119–2141.
- [7] J.C. Butcher, Z. Jackiewicz, Construction of high order diagonally implicit multistage integration methods for ordinary differential equations, *Appl. Numer. Math.* 27 (1998) 1–12.
- [8] J.C. Butcher, Z. Jackiewicz, H.D. Mittelmann, Nonlinear optimization approach to construction of general linear methods of high order, *J. Comput. Appl. Math.* 81 (1997) 181–196.
- [9] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, New York, 1988.
- [10] C. Chiu, D.A. Kopriva, An optimal Runge–Kutta method for steady-state solutions of hyperbolic systems, *SIAM J. Numer. Anal.* 29 (1992) 425–438.
- [11] R. Clayton, B. Engquist, Absorbing boundary conditions for acoustic and elastic wave equations, *Bull. Seismol. Soc. Amer.* 67 (1977) 1524–1540.
- [12] B. Fornberg, *A Practical Guide to Pseudospectral Methods*, Cambridge University Press, Cambridge, 1996.
- [13] E. Hairer, S.P. Nørsett, G. Wanner, *Solving Ordinary Differential Equations I. Nonstiff Problems*, Springer-Verlag, Berlin, 1991.
- [14] R.L. Higdon, Numerical absorbing boundary conditions for the wave equation, *Math. Comp.* 49 (1987) 65–90.
- [15] Z. Jackiewicz, R.A. Renaut, A note on stability of pseudospectral methods for wave propagation, manuscript, Arizona State University, 1999.
- [16] Z. Jackiewicz, R. Vermiglio, General linear methods with external stages of different orders, *BIT* 36 (1996) 688–712.
- [17] Z. Jackiewicz, R. Vermiglio, M. Zennaro, Variable stepsize diagonally implicit multistage integration methods for ordinary differential equations, *Appl. Numer. Math.* 16 (1995) 343–367.
- [18] Z. Jackiewicz, R. Vermiglio, M. Zennaro, Regularity properties of multistage integration methods, *J. Comput. Appl. Math.* 87 (1997) 295–302.
- [19] J.D. Lambert, *Computational Methods in Ordinary Differential Equations*, Wiley, Chichester, 1973.
- [20] J.L. Mead, *Computational methods for aeroacoustics*. Ph.D. Thesis, Department of Mathematics, Arizona State University, Tempe, AZ, August 1998.
- [21] R.A. Renaut, J. Fröhlich, A pseudospectral Chebyshev method for the 2-D wave equation with domain stretching and absorbing boundary conditions, *J. Comput. Phys.* 124 (1996) 324–336.
- [22] S.C. Reddy, L.N. Trefethen, Lax-stability of fully discrete spectral methods via stability regions and pseudo-eigenvalues, *Comput. Methods Appl. Mech. Engrg.* 80 (1990) 147–164.

- [23] P.J. van der Houwen, B.P. Sommeijer, Phase-lag analysis of implicit Runge–Kutta methods, *SIAM J. Numer. Anal.* 26 (1989) 214–229.
- [24] J. Van Wieren, Using diagonally implicit multistage integration methods for solving ordinary differential equations. Part 1: Introduction to explicit methods, NAWCWPNS TP 8340, Naval Air Warfare Center Weapons Division, China Lake, CA, January 1997.
- [25] J. Van Wieren, Using diagonally implicit multistage integration methods for solving ordinary differential equations. Part 2: Implicit methods, NAWCWPNS TP 8356, Naval Air Warfare Center Weapons Division, China Lake, CA, August 1997.
- [26] J. Van Wieren, Using diagonally implicit multistage integration methods for solving ordinary differential equations. Ph.D. Thesis, Department of Mathematics, Arizona State University, Tempe, AZ, July 1997.