Convergence acceleration of preconditioned indefinite systems for second order elliptic boundary value problems

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Abstract

Using the mixed finite element discretization of a second-order elliptic boundary value problem, such as the Stokes problem, we obtain a discretized system of saddle point structure. The system matrix, usually of large dimension, is symmetric, nonsingular, indefinite and ill-conditioned. Based on the determination of the optimal spectral radius for the iteration matrix at the elemental level, an effective preconditioning for the solution of an indefinite linear system is determined. The rate of convergence of the method is independent of the mesh parameter and by an appropriate choice of parameters can be made arbitrarily small.

KEYWORDS: Stokes problem, finite elements, convergence, preconditioning

1 Introduction

Using the mixed finite element discretization of a second-order elliptic boundary value problem, such as the Stokes problem, we obtain a discretized system,

$$A \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} M & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \Phi, \tag{1}$$

of saddle point structure [?]. Matrix $M \in R^{m \times m}$ is symmetric positive definite (SPD), and $B \in R^{n \times m}$, $n < m$, is of full rank. In practice the coefficient matrix A is symmetric, nonsingular, indefinite and ill-conditioned. Further, the dimension of the system is usually large, $n + m >> 0$. Hence an iterative solution to (**??**) is sought. However, because the coefficient matrix A is indefinite and ill-conditioned, it is difficult it construct an effective iterative method, unless preconditioning is applied. Suppose that a preconditioner N can be constructed, such that the resultant coefficient matrix $N^{-1}A$ is well conditioned. Then if $N^{-1}A$ is SPD, the standard preconditioned conjugate gradient method can be applied to solve (**??**), (cf. [?] [?]).

Based on the determination of the optimal spectral radius for the iteration matrix at the elemental level, an effective preconditioning for the indefinite system (**??**) is obtained. The rate of convergence of the method is independent of the mesh parameter and it can

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be made arbitrarily small. Hence the proposed method should converge faster than that introduced by Ewing et al. [?], for which a different elemental preconditioning was used.

Here we present the approach for the Dirichlet Problem

$$\left\{ \begin{array}{rcll} -\nabla \cdot (k(x)\nabla p) & = & -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(k_{ij}(x)\frac{\partial p}{\partial x_j}) = f & \text{in } \Omega \in R^d, \\ p & = & 0 & \text{on } \partial\Omega \end{array} \right. \quad (2)$$

where f is given, and the coefficient matrix defined by $k(x) = (k_{ij}(x))_{d\times d}$ is symmetric and uniformly positive definite in the closed domain $\bar{\Omega}$.

2 Preconditioning the indefinite system and the structure of the iterative method

We consider the two dimensional case for which there are no mixed partial derivatives in the Dirichlet problem. Then if we partition the unknown values of U into a block structure $U = (U_1^T, U_2^T)^T$, matrix M is block diagonal

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix}. \quad (3)$$

We choose the preconditioner

$$N = \begin{pmatrix} \tilde{M} & lB^T \\ lB & 0 \end{pmatrix}, \qquad l > 0, \quad (4)$$

dependent on the relaxation parameter l, where $\tilde{M}$, which can be taken block diagonal, is required to be positive definite and invertible. Clearly, N^{-1} exists, because B has full rank.

For the system $Ax = \Phi$, an iterative method can be constructed as follows:

$$x^{(k+1)} = (I - \theta N^{-1}A)x^{(k)} + \theta N^{-1}\Phi, \qquad k = 0, 1, \ldots \quad (5)$$

where $x^{(0)}$ is given, $\theta > 0$ is a second relaxation parameter and for system (**??**) we have

$$x^{(k)} = \begin{pmatrix} U^{(k)} \\ P^{(k)} \end{pmatrix}, \qquad \Phi = \begin{pmatrix} 0 \\ -f \end{pmatrix}.$$

Note that for $\theta = l = 1$, this iteration reduces to that in [?]. In practice $x^{(k+1)}$ is obtained by solving the linear system

$$Nx^{(k+1)} = \theta\Phi + (N - \theta A)x^{(k)}. \quad (6)$$

The convergence and the speed of the convergence of the algorithm are determined by the spectral radius, $\rho(R)$, of the iteration matrix $R = (I - \theta N^{-1}A)$, which depends on the eigenvalues of $N^{-1}A$ (cf. [?]), and the choice of θ. Proofs of the following results which detail the properties of the eigenvalues of $N^{-1}A$ are given in Ding [?] and follow as in [?]. For arbitrary matrix A, we denote by $\sigma(A)$ and $\kappa(A)$ the spectrum and condition of A, respectively, where $\kappa(A) = \lambda_{\max}/\lambda_{\min}$ and $\lambda_{\max}$, $\lambda_{\min}$ are the maximum and minimum eigenvalues of A, respectively.

Lemma 2.1 *Suppose that $M \in R^{m\times m}$ is SPD, B is of full rank, $n < m$, and $\tilde{M}$ is positive definite then*

a) $0 < \mu_{min} \le \mu \le \mu_{max},\ \mu \in \sigma(\tilde{M}^{-1}M)$

b) $0 < \mu_{min} \le \lambda = \frac{1}{l} \le \mu_{max},\ \lambda \in \sigma(N^{-1}A)$

c) $\rho(N^{-1}A) \le \rho(\tilde{M}^{-1}M)$

d) $\kappa(N^{-1}A) \le \kappa(\tilde{M}^{-1}M)$.

Lemma 2.2 *For the generalized conjugate gradient method applied to the preconditioned system*

$$\theta N^{-1}A\begin{pmatrix} U \\ P \end{pmatrix} = \theta N^{-1}\begin{pmatrix} 0 \\ -f \end{pmatrix}$$

the rate of linear convergence is bounded by $\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$*, where*

$$\kappa = \kappa(\theta N^{-1}A) = \kappa(N^{-1}A) \le \kappa(\theta\tilde{M}^{-1}M) = \kappa(\tilde{M}^{-1}M). \tag{7}$$

Hence to increase the convergence speed of the iterative method (**??**), it suffices to make $\rho(I - \theta\tilde{M}^{-1}M)$ as small as possible. Because $\tilde{M}^{-1}M$ is block diagonal, it suffices to minimize $\rho(I - \theta\tilde{M}_{ii}^{-1}M_{ii})\}, i = 1, 2$. To accomplish this, we first identify the mass matrix M.

2.1 Construction of the Mass Matrix M

Suppose that M is obtained using standard parallelepiped finite elements for a partitioning T_h of domain Ω, and Simpson's rule is applied for the quadrature [**?**]. Then matrices M_{ii}, $i = 1, 2$, are also block diagonal, and identical, with entries determined by the individual elements $e \in T_h$. Then the preconditioner $\tilde{M}$ is structured consistently with M with elementwise preconditioner $\tilde{M}^e$, $e \in T_h$, given by

$$\tilde{M}^e = \begin{pmatrix} \tilde{M}^e_{11} & 0 \\ 0 & \tilde{M}^e_{22} \end{pmatrix}, \qquad \tilde{M}^e_{ii} = \omega^e_i \check{M}^e_{11}, \qquad \check{M}^e_{11} = \begin{pmatrix} \check{M}^e_0 & 0 \\ 0 & \check{M}^e_0 \end{pmatrix}, \tag{8}$$

where the scaling parameters ω^e_1 and ω^e_2 are chosen to minimize $\rho(I - \tilde{M}^{-1}M)$. Because $\tilde{M}$ is required to be P.D. so are the $\tilde{M}^e$. Again, by block diagonality,

$$\rho(I - \theta(\tilde{M})^{-1}M) = \max\{\max_{e\in T_h} \rho(I - \frac{\theta}{\omega^e_1}(\check{M}^e_{11})^{-1}M^e_{11}), \max_{e\in T_h} \rho(I - \frac{\theta}{\omega^e_2}(\check{M}^e_{11})^{-1}M^e_{11})\}. \tag{9}$$

Hence the local parameter θ/ω^e, $\omega^e_1 = \omega^e_2 = \omega^e$, should be chosen such that

$$\rho\Big(I - \frac{\theta}{\omega^e}(\check{M}^e_{11})^{-1}M^e_{11}\Big)$$

is minimal. This is accomplished using the same method as in [**?**]:

Theorem 2.1 *Suppose that matrix $\check{M}_{11}^e$ is positive definite and let $0 < \mu_o^e < \mu_1^e$ be the eigenvalues of matrix $(\check{M}_{11}^e)^{-1}M_{11}^e$ for each finite element $e \in T_h$. Then with the choice*

$$\frac{\theta}{\omega^e} = \frac{2}{\mu_1^e + \mu_o^e} \tag{10}$$

*the iterative method (**??**) is convergent and the rate of linear convergence is bounded,*

$$\rho(I - \theta N^{-1}A) \le \max_{e\in T_h} \frac{\mu_1^e - \mu_o^e}{\mu_1^e + \mu_o^e} < 1. \tag{11}$$

3 Convergence Acceleration

In order to accelerate the convergence of the iterative method (**??**), we choose the local preconditioning matrix $\check{M}_0^e$, different to that in [**?**], to be

$$\check{M}_0^e = \begin{pmatrix} m_{11} & 0 \\ \gamma^e m_{12} & m_{22} \end{pmatrix}, \tag{12}$$

where $\gamma^e \ge 0$ and m_{ij} are elements of matrix M_0^e, $m_{12} = m_{21}$. Then the eigenvalues of $(\check{M}_0^e)^{-1}M_0^e$ are the solutions of

$$\mu^2 + (\gamma^e t^e - 2)\mu + (1 - t^e) = 0 \tag{13}$$

where $t^e = m_{12}^2/m_{11}m_{22}$, and the choice of γ^e depends on t^e.

Lemma 3.1 *For $k(x)$ continuous and uniformly positive definite on $\bar{\Omega}$, and coefficients m_{ij} determined by Simpson's rule for each element $e \in T_h$, then*

i) $0 < t^e < 1$

ii) in the limit as the grid is refined $t^e \to 1/1295$.

The proof of this result, given in Ding [**?**], depends on identification of coefficients m_{ij}. From Lemma 3.1 we deduce that the roots of (13) are real for $0 < \gamma^e < \frac{2}{t^e}[1 - \sqrt{1 - t^e}]$. Moreover, the rate of linear convergence and optimal choice for $\frac{\theta}{\omega^e}$ can be determined.

Lemma 3.2 *Suppose matrix M is P.D. Then, for $0 < \gamma^e < \frac{2}{t^e}[1 - \sqrt{1 - t^e}]$, $e \in T_h$,*

i) $\frac{\mu_1^e - \mu_0^e}{\mu_1^e + \mu_0^e} = \frac{[(\gamma^e t^e)^2 - 4\gamma^e t^e + 4t^e]}{2 - \gamma^e t^e}^{1/2} = \psi(\gamma^e) \le \sqrt{t^e}$

ii) $\frac{\theta}{w^e} = \frac{2}{2 - \gamma^e t^e} = \phi(\gamma^e t^e) > 1.$

The proof is immediate by showing ψ is decreasing and ϕ is increasing for γ^e in the range given and $0 < t^e < 1$.

Theorem 3.1 *Suppose M is P.D., and let $t = \max_e t^e$. Then*

i) iteration (5) converges independently of the mesh h

ii) the rate of linear convergence is bounded

$$\rho(I - \theta N^{-1}A) \leq \sqrt{t} \leq \frac{d^2}{(d+34)^2} < 1,$$

where $d = \max_{x\in\bar{\Omega}} k(x)/\min_{x\in\bar{\Omega}} k(x)$.

iii) $\rho(I - \theta N^{-1}A)$ *can be chosen arbitrarily small, e.g., for* $\gamma^e = 1 + \frac{t^e}{4}$

$$\rho(I - \theta N^{-1}A) < \frac{t^{3/2}}{\sqrt{8}} + \text{ higher order terms in } t$$

and the optimal ω^e *is given by*

$$\omega^e = \frac{\theta(2 - \gamma^e t^e)}{2} = \theta\left(1 - \frac{t^e}{2} - \frac{{t^e}^2}{8}\right)$$

iv) For $\gamma^e = 1 < \frac{2}{t^e}[1 - \sqrt{1 - t^e}]$

$$\rho(I - \theta N^{-1}A) < \frac{t}{2-t} < \sqrt{t}$$

and $w^e = \frac{\theta}{2}(2 - t^e)$.

Note, for $\theta = l = 1$, the bound on the rate of linear convergence is less than that for the method in [**?**], $d^2/(d+34)^2 < d/(1+d)$. Hence, theoretically, the proposed preconditioning is more efficient. Appropriate choices of θ and l, and numerical results evaluating the proposed preconditioning will be discussed in [**?**].

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