Absorbing Boundary Conditions

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Absorbing Boundary Conditions are boundary procedures that are applied at the artificial numerical boundaries of a computational domain to miminize or eliminate the spurious relections at these boundaries which occur in the simulations of wave propagation phenomena. They may also be called **non–reflecting boundary conditions** or **radiating boundary conditions**. There are different ways to design appropriate boundary procedures, here just one approach is considered. References to other procedures are provided.

Suppose that the solution of the **acoustic wave equation**,

$$u_{tt} = c^2(u_{xx} + u_{yy}), \tag{1}$$

on an unbounded domain is desired. The solutions of this equation are plane waves, $u = u(x, y, t) = e^{i(\omega t+\xi x+\eta y)}$, where ω is the frequency and ξ and η are the spatial wave numbers, which travel in all directions. Computationally the domain is of finite size and boundaries are introduced. Therefore waves incident at the boundary will allow for non-physical reflections back into the domain. It is these reflections that are to be minimized. A technique that has proven successful, is the application of absorbing boundary conditions which have been derived from approximations to a one-way wave equation (OWWE) at the boundary [3],[5],[6], [10],[8].

For (1) a wave with wave numbers ξ, η travels at the velocity $(c_x, c_y) = c(-\xi/\omega, -\eta/\omega) = c(-\cos\theta, -\sin\theta)$, where θ is the angle measured counterclockwise from the negative x-axis; waves with $|\theta| < 90^\circ$ travel to the left, $c_x < 0$, and waves with $|\theta| > 90^\circ$ travel to the right, $c_x > 0$. At an artificial numerical boundary $x = 0$, waves should travel to the left; $c_x \le 0$, or equivalently ξ and ω are of the same sign. However, solutions of (1) satisfy the dispersion relation

$$\omega^2 = c^2(\xi^2 + \eta^2), \tag{2}$$

and ξ and ω are related by the relationship

$$\xi = \pm\frac{\omega}{c}\sqrt{1-\left(\frac{\eta c}{\omega}\right)^2}. \tag{3}$$

For the positive root ξ and ω are of the same sign and waves travel to the left only. Therefore, (3) with the positive sign represents the dispersion relation for the appropriate boundary condition. This dispersion relation, however, corresponds to a **pseudo-differential** operator. To obtain a partial differential equation for waves which travel only to the left it is necessary to approximate the square root in (3). The resulting equation is called a **one-way wave equation**. Engquist and Majda [4] designed the **paraxial** OWWE equations based on **Padé** approximations to $\sqrt{1-s^2}$, $s = \frac{\eta c}{\omega}$. For the simple Padé approximation, $1 - \frac{1}{2}s^2$, the OWWE used as a boundary condition at $x = 0$ is $cu_{xt} = u_{tt} - \frac{1}{2}c^2 u_{yy}$. A general reference on the derivation of OWWEs using the rational approximants $r(s)$ is given in [5]. Lindman [6] adopted a similar approach which yields a system of equations at the boundary that can very easily be augmented to allow for approximations of higher order.

The potential effectiveness of the OWWE is measured by its **reflection coefficient**

$$(4) \quad R(s) = \left| \frac{r(s) - \sqrt{1-s^2}}{r(s) + \sqrt{1-s^2}} \right|, \qquad s \in [-1, 1].$$

For minimal reflection $R(s) << 1$ is desirable. The implementation of the boundary condition can not ensure that the reflection is minimal unless an appropriate numerical approximation can be determined. In particular, a stable **finite difference** approximation to the OWWE is also required. Renaut [8] provides a standard approach by which the differential equation at the boundary can be discretised and give a stable implementation. Numerical implementations for the solution of the acousic wave equation and the reflection coefficients are studied in [8]. For electromagnetics the standard approach uses the **Mur** boundary conditions, [7]. In [12] high order boundary conditions are tested for the numerical solution of **Maxwell's equations**.

Finite difference solutions of partial differential equations are usually local in space because only a few grid points on the computational grid are employed to derive approximations to the underlying partial derivatives in the equation. To obtain more accurate solutions either higher–order approximations can be derived or global solution techniques can be considered. The higher-order finite difference approximations tend to make the design of boundary conditions more difficult because grid points near the boundary are not automatically defined. To absorb incident waves at the computational boundary one approach uses a **damping region**. In this case the computational domain is increased in size but the solution is accepted only on the smaller domain. Within the damping region the wave is progressively damped to zero, [11]. The method is successful but suffers from the disadvantage of a computational overhead induced by the damping region, which is considerable for three dimensions.

Global approximations for partial differential equations as in **pseudospectral methods** (PS methods) are increasingly popular. The PS methods use global interpolation to approximate the unknown function and its derivatives on the computational domain. Implementation of boundary operators is not immediate, although damping regions have been used successfully, [2]. Recently, absorbing boundary conditions derived from the OWWEs have also been successfully implemented for PS methods , [9].

The **Perfectly Matched Layer** (PML) introduced by Berenger, [1], for Maxwell's equations involves the application of a non-physical absorbing material adjacent to the computational boundary. The method is implemented by splitting certain field components in the PML region into subcomponents which can be perfectly absorbed by the PML material. Numerical tests report that this approach is superior to the use of OWWEs for electromagnetics.

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