On the Courant-Friedrichs-Lewy Condition Equipped with Order for Hyperbolic Differential Equations

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Abstract

In 1928 Courant, Friedrichs and Lewy showed a necessary condition for convergence of a difference scheme for solving a hyperbolic differential equation. This condition states that the numerical domain of dependence has to include the exact domain of dependence. Applying this argument to the linear advection equation, where the exact domain of dependence is given by the characteristic line, says that an explicit difference stencil must have at least one stencil point on both sides of the characteristic line through the stencil point on the newest time level. In 1985 this result has been generalized to explicit and implicit, normalized, two time level difference schemes in the form that a stable scheme with local error order p must have at least $\lceil p/2 \rceil$ stencil points on each side of the characteristic line. It is conjectured that this statement is correct for multistep difference schemes. In the present paper we consider three time level schemes and can show the correctness of the conjecture for schemes of optimal order $p =$ number of stencil points -2 where the Courant number μ satisfies $-1/2 < \mu < 0$. Moreover, the conjecture can be shown for all schemes with a convex increasing stencil using a conjecture on the so called order star of the scheme. In particular this result applies to implicit schemes.

1 Introduction

When using linear difference schemes for initial boundary value problems of linear hyperbolic partial differential equations it has been shown that it is essential to study the schemes when applied to the pure Cauchy problem for the scalar advection equation

$$(1)\qquad \begin{aligned} u_t(t,x) &= c\,u_x(t,x),\ x \in \mathbb{R},\ t \geq 0 \\ u(0,x) &= u_0(x) \text{ given }, \end{aligned}$$

see e.g. [4], [2]. As indicated in the lecture of VanLeer in this conference conditioning of the Euler equations of gas dynamics is a way of preparing these nonlinear equations such that they behave more like linear equations and hence our linear analysis sheds some light even on this nonlinear case. However it should be noted that the schemes we are applying to (1) in the present article are linear only. We consider the following class of multistep ($(k+1)$-time-level) difference schemes of the form

$$(2)\qquad \sum_{i=0}^{k} \sum_{j=-r_i}^{s_i} a_{ij}\, u_{n+i,m+j} = 0\ ,\ n = 0,1,2,...,m \in \mathbb{Z}\ .$$

Here $u_{n,m}$ is supposed to approximate the exact solution of (1) $u(n\Delta t, m\Delta x)$ where Δt and Δx are the uniform stepsizes in time and space. Schemes of form (2) can be used for more general systems of partial differential equations either as an interior scheme or as a boundary scheme. As an interior scheme one always assumes that such a scheme is stable in the von Neumann sense when applied to the pure Cauchy problem (1), see e.g. [4]. Observe that Goldberg and Tadmor [2], [3] gave practical sufficient conditions for stability of global schemes for initial boundary value problems. These conditions entail, among others, that the boundary scheme also has to be stable in the sense mentioned above for the interior scheme. Whether the scheme is used in the interior or at the boundary it has to represent the equation (1) with a certain accuracy, that is a smooth solution of (1) has to satisfy the difference formula (2) up to terms of the form $O(\Delta x^{p+1})$. Here we have assumed that $c\ \Delta x/\Delta t$ is kept constant. p is called the order of accuracy. In this paper we investigate the interplay between the accuracy and the stability of three-time-level schemes. The stability requirement gives a bound for the highest possible order a scheme with a prescribed stencil can have.

The above mentioned interplay between the error order and the stability of the scheme is well understood for one-step formulas ($k = 1$), [10]. For multistep schemes partial results have been given in [6], [8], [12], [13]. These are in agreement with the conjecture which states that the order of accuracy p of a stable scheme satisfies the inequality

$$(3)\qquad p \leq 2\ \min\{R, S\}\ .$$

Here R and S denote the number of stencil points to the left and right, respectively, of the characteristic line through the point $(n\,\Delta t, m\,\Delta x)$. For explicit schemes, $r_k = s_k = 0$, this characteristic line represents the exact domain of dependence. If the scheme is implicit, i.e. at least two coefficients of a_{kj} are nonzero, this line has a similar function if the scheme is normalized in a fashion to be in agreement with the Wiener-Hopf theory on solutions of doubly infinite linear systems, see Section 2 or [5], [10]. The conjecture (3) says that a stable scheme of order p needs at least $\lceil p/2 \rceil$ stencil points on each side of the characteristic line described above. ($\lceil \alpha \rceil$ denotes the smallest integer which is not smaller than α). Observe that for explicit schemes of order $p \geq 1$ the conjecture says that on either side of the characteristic line one needs at least one stencil point. This can be interpreted as the statement of Courant-Friedrichs-Lewy that the numerical domain of dependence has to include the exact domain of dependence in order for a scheme to be convergent, [1].

The conjecture (3) has been proved in [10] for one-step schemes. In [6] and [8] many examples in support of the conjecture for multistep schemes, $k > 1$, were given. In [7] a stable $(k+1)$-time level scheme has been given which shows that if the barrier (3) would be correct it would be a sharp bound. In the present article we follow the lines of [12] where the conjecture (3) has been proved for a small class of explicit three-time-level schemes. In order to prove the result for all explicit and implicit three-time-level schemes we have to resort to another conjecture, Conjecture 5.1. In the important case of schemes of maximal order this conjecture and hence (3) can be proved at least for **convex schemes** with an increasing stencil. This proof could only be done for $-1/2 < \mu < 0$. However we do believe that we shall be able to extend it to $0 < \mu < 1/2$.

In this article we can, due to space limitations only highlight the ideas of the proofs. For details we refer to [9]. In Section 2 we give the basic definitions and formulate the main theorem in Section 3. To start explaining the proof we describe the order star in Section 4 and give in Section 5 the important bounds for the multiplicities of the order star components. There we also explain Conjecture 5.1. In the last section we outline the proof and give an interpretation of the result.

2 Order, stability and normalization of schemes

In this article we shall restrict ourselves to three-time-level schemes, i.e. schemes of form (2) with $k = 2$ for solving the Cauchy-problem (1). We assume that while Δx and Δt tend to zero the Courant number $\mu = c\Delta t/\Delta x$ is kept constant. The coefficients a_{ij} are real and depend in general on μ. Further $r_i,\ s_i \in \mathbb{Z}$ with $r_2 \geq 0,\ s_2 \geq 0$ and $-r_i \leq s_i,\ i = 0,1$ and $a_{i,-r_i} \neq 0$, $a_{i,s_i} \neq 0$ for $i = 0,1,2$. If $r_2 = s_2 = 0$ a scheme is said to be **explicit**. Otherwise it is called **implicit**.

The interplay between accuracy and stability can best be handled mathematically by making a discrete Fourier Transform in space of $u_{n,m}$ in (2). One obtains

$$(4) \qquad \sum_{i=0}^{2} a_i(e^{i\Delta x\xi})\,\widetilde{u}_{n+i}(\xi) = 0,\ n = 0,1,2,...$$

where $\widetilde{u}_n(\xi)$ is the Fourier Transform of $u_{n,m}$ at $t = \Delta t \cdot n$ and

$$(5) \qquad a_i(z) = \sum_{j=-r_i}^{s_i} a_{ij}\, z^j,\ i = 0,1,2,\ .$$

In order to be able to solve (2) or (4) for the values on the new time-level $t = (n+2)\Delta t$ we need

$$(6) \qquad a_2(z) \neq 0 \ \text{ for } \ |z| = 1\ .$$

Given a scheme with a stencil the numbers r_i, s_i are a priori not uniquely defined since one could replace these by $r_i' = r_i + 1,\ s_i' = s_i - 1$ without changing formula (2). In order to make the definition unique we observe that $z^{r_2}a_2(z)$ has exactly $r_2 + s_2$ zeros, none of which are on the unit circle. Hence we define r_2 and s_2 according to the following **normalization condition**.

Normalization condition

$$(7)\qquad \begin{aligned} r_2 &= \text{number of zeros of } a_2(z) \text{ with } |z|<1 \\ s_2 &= \text{number of zeros of } a_2(z) \text{ with } |z|>1\ . \end{aligned}$$

Observe that while this normalization seems arbitrary for the pure Cauchy-problem it is by the Wiener-Hopf theory a necessary requirement when solving initial boundary value problems, see [5]. Furthermore by the normalization condition (7) the characteristic line mentioned in the introduction is uniquely defined even for implicit schemes.

(4) is a three-term-recurrence relation. Hence its solutions are bounded if the roots of the polynomial

$$a_2(e^{i\Delta x\xi})\, w^2 + a_1(e^{i\Delta x\xi})\, w + a_0(e^{i\Delta x\xi})$$

have modulus not exceeding 1 and those of modulus 1 are simple. Therefore we introduce the **characteristic function**

$$(8)\qquad \Phi(z,w) := a_2(z)\, w^2 + a_1(z)\, w + a_0(z)$$

and say that the scheme (2) is **stable** if

$$(9)\qquad \left.\begin{array}{r} \Phi(z,w)=0 \\ |z|=1 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} |w|\le 1 \text{ and if } |w|=1 \\ \text{then } w \text{ is a simple root}\ . \end{array}\right.$$

As scheme has **error order** p if for any smooth solution $u(t,x)$ of (1) we have

$$(10)\qquad \sum_{i=0}^{2}\sum_{j=-r_i}^{s_i} a_{ij}\, u(t+i\Delta t, x+j\Delta x) = C\,\frac{\partial^{p+1}}{\partial x^{p+1}}\, u(t,x)(\Delta x)^{p+1} + O((\Delta x)^{p+2})$$
$$\text{if } \Delta x \to 0 \text{ and } \mu = \text{constant}\ .$$

We are interested in schemes with a positive order, hence we always assume that

$$(11)\qquad \Phi(1,1) = \sum_{i=0}^{2}\sum_{j=-r_i}^{s_i} a_{ij} = 0\ .$$

Hence $\Phi(1,w)$ has a root which is 1. If the scheme is stable then by (9) this root is a simple one. This implies that the **algebraic function** $w(z)$ defined implicitly by

$$(12)\qquad \Phi(z, w(z)) \equiv 0\ ,$$

has a branch $w_1(z)$ which is analytic in a neighborhood of $z=1$ and satisfies $w_1(1)=1$. One can easily show that a stable scheme has order p if and only if this branch satisfies

$$(13)\qquad z^{\mu} - w_1(z) = O((z-1)^{p+1}) \ \text{ as } \ z\to 1\ ,$$

see [8], [13]. This implies that one can assume without loss of generality that Φ is irreducible.

Observe that (10) represents linear conditions for the coefficients a_{ij}. Let I be the set of indices

$$I = \{(i,j)\in \mathbb{Z}\times\mathbb{Z} \,|\, 0\le i\le 2,\ -r_i\le j\le s_i\}\ .$$

A scheme is said to be **regular** if no characteristic line passes through two stencil points. For regular stencils the system of equations for the a_{ij} is regular for the highest possible order

$p = |I| - 2$ and hence we always have a unique scheme of highest error order for a given regular stencil. Of course such a scheme need not to be stable.

Due to the techniques used in the proofs one can only hope to prove the conjecture (3) for **convex** schemes. A scheme is called **convex** if the convex hull of the stencil contains all extreme points $(-r_i, i)$, (s_i, i), $i = 0, 1, 2$ of the stencil. For illustration the stencils in Fig. 1a),b) are not convex since the grid point $(-1, 1)$ does not lie on the convex hull. The scheme of Fig. 1c) however is convex. In the present paper we restrict ourselves to schemes with a convex and increasing stencil. A scheme is said to have an **increasing stencil** if r_i and s_i are non-decreasing with respect to i. Hence the stencil of Fig. 1c) is not increasing but convex. The stencil of Fig. 1d) however is convex and increasing. For two step-schemes the condition to be convex and have an increasing stencil is equivalent to the condition

$$(14) \qquad \begin{cases} 0 \;\leq\; r_0 - r_1 \;\leq\; r_1 - r_2 \\ 0 \;\leq\; s_0 - s_1 \;\leq\; s_1 - s_2\ . \end{cases}$$

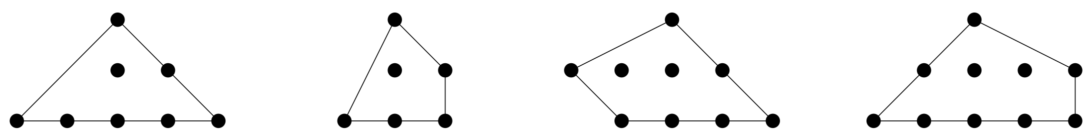

Fig. 1: Stencils and their convex hull

From a computational point of view these schemes seem the most interesting class of schemes to consider. While we believe the conjecture (3) can be proved for convex schemes with an increasing stencil the actual proof in [9] is only done for a class of schemes which is slightly smaller. We shall indicate this difference in Section 4, Remark 4.2.

3 Main results

For two step schemes we define exactly the numbers R and S by

$$(15a) \qquad R = R_0 + R_1 + r_2,\ S = S_0 + S_1 + s_2\ ,$$

where R_i, S_i are defined as follows

$$(15b) \qquad R_1 = \begin{cases} 0 & \text{if} \quad \mu < -r_1 \\ \lfloor r_1 + \mu \rfloor + 1 & \text{if} \quad -r_1 < \mu < s_1 \\ r_1 + s_1 + 1 & \text{if} \quad \mu > s_1 \end{cases},\ S_1 = r_1 + s_1 + 1 - R_1$$

$$(15c) \qquad R_0 = \begin{cases} 0 & \text{if} \quad \mu < -r_0 \\ \lfloor r_0 + 2\mu \rfloor + 1 & \text{if} \quad -r_0 < 2\mu < s_0 \\ r_0 + s_0 + 1 & \text{if} \quad 2\mu > s_0 \end{cases},\ S_0 = r_0 + s_0 + 1 - R_0\ .$$

Here $\lfloor \alpha \rfloor$ denotes the largest integer not exceeding α.

Theorem 3.1 *Let a convex scheme (4) with an increasing stencil be normalized and have a Courant number μ satisfying $0 < |\mu| < 1/2$. If the scheme is stable, then the order p of the scheme is bounded by*

$$p \leq 2 \min\{R, S\} . \tag{16}$$

□

Remark 3.2

a) In [12] the bound (16) was proved for a small subclass of explicit schemes of type (4). In the present paper we generalize it, making use of a conjecture introduced in Section 5, for a class of explicit and implicit schemes of type (4) which are convex and have an increasing stencil.

b) In [9] we prove the conjecture for schemes of maximal order $p = |I| - 2$ and $-1/2 < \mu < 0$ and give here a brief hint of the proof in Remark 5.2.

c) As indicated the current proofs are valid only for a slightly smaller class than the convex schemes with increasing stencil, see Remark 4.2. However this smaller class includes all schemes with

$$\begin{cases} 0 \leq r_0 - r_1 < r_1 - r_2 \\ 0 \leq s_0 - s_1 < s_1 - s_2 \end{cases} . \tag{17}$$

d) The result (16) can be extended to $|\mu| > \frac{1}{2}$ by making use of the following transformation. Assume that a stable scheme is represented by $\Phi(z, w)$, where $w(z)$ approximates z^μ in a neighbourhood of the point $z = 1, w = 1$. Then we consider the scheme represented by the characteristic function

$$\widetilde{\Phi}(z, u) = z^2 \, \Phi(z, \tfrac{u}{z}) .$$

Since $u = zw$, the new scheme is stable and approximates $z^{\widetilde{\mu}} = z^{\mu+1}$ with the same order as the original scheme. The stencil undergoes the following transformations:

$$\widetilde{r}_1 = r_1 - 1, \quad \widetilde{s}_1 = s_1 + 1, \quad \widetilde{r}_0 = r_0 - 2, \quad \widetilde{s}_0 = s_0 + 2 .$$

4 The algebraic function w and order stars

The algebraic function $w(z)$ defined by Φ as a solution of (12) can either be viewed as a double valued function, consisting in general for each z of two values $w_1(z)$ and $w_2(z)$ or we can interpret $w(z)$ as a single valued function on the compact Riemann surface,

$$M = \{(z, w) \in \bar{\mathbb{C}} \times \bar{\mathbb{C}} \mid \Phi(z, w) = 0\} .$$

M is a connected set on which w is analytic except for points (z_i, ∞) where w has a pole. Since in the order star technique poles play an important role we need to know where these occur.

Proposition 4.1 *The function w has a pole at every point where $a_2(z) = 0$. In view of normalization (7), there will be exactly r_2 poles in $|z| < 1$ and s_2 in $|z| > 1$. Moreover if there is no branch point of $w(z)$ at $z = 0$ then the branches $w_i(z)$ have the following expansions*

$$\begin{aligned} w_1(z) &= z^{-(r_1-r_2)}(c_0 + c_1\, z + c_2\, z^2 + \dots) \\ w_2(z) &= z^{-(r_0-r_1)}(d_0 + d_1\, z + d_2\, z^2 + \dots) \end{aligned} \tag{18}$$

at $z = 0$.

Remark 4.2

At this point we can explain the small set for which our proof has not been carried through completely. In the proof we assume that there is no branch point at $z = 0$. Branch points are not really special points on a Riemann surface however in the order star technique used here $z = 0$ is a very special point. If $r_1 - r_0 = r_2 - r_1$ it can in rare cases happen that there is a branch point at $z = 0$. However then a slight perturbation of the coefficients a_{ij} will move the branch point away from zero. However a stencil of form given in Fig. 1b) always has a branch point at $z = 0$ even if one perturbs the coefficients a_{ij} slightly. This is why we have excluded these type of stencils by requesting convexity. Observe that condition (17) automatically ensures that there is no branch point at zero. In the following we shall always assume that $z = 0$ and $z = \infty$ are not branch points of w.

At this point it is possible to explain why one requests that the stencil is increasing. The zeros of w are at points $(z_i^\circ, 0)$ with $a_0(z_i^\circ) = 0$ and at $z = 0$ or $z = \infty$ if the stencil is not increasing. For example from (18) we see that a stencil of the form given in Fig. 1c) has a zero at $z = 0$. Since we currently have no definite proofs to locate the zeros we cannot adapt our proofs to work with zeros. Hence the shape of the stencil which is related to zeros cannot be efficiently treated and therefore we have to restrict ourselves to increasing stencils.

Definition 4.3 The set

$$\Omega = \{(z, w) \in M \,\Big|\, |z^{-\mu} w| > 1\}$$

is said to be the **order star**. □

We shall frequently work with the function

$$\varphi(z, w) = z^{-\mu} w, \ (z, w) \in M$$

which is multivalued on M due to the factor $z^{-\mu}$. Ω^c denotes the complement of Ω, i.e. $\Omega^c = M \backslash \Omega$. Because the coefficients a_{ij} are real Ω is symmetric with respect to the real axis. As in the classical order star technique introduced in [14] the following two lemmata express stability and accuracy in terms of the order star.

Lemma 4.4 *(Stability)*

If a scheme is stable then

$$\Omega \cap \{(z, w) \in M \,\Big|\, |z| = 1\} = \emptyset\,.$$ □

Lemma 4.5 *(Order)*

A stable scheme of form (2) has order p if and only if at the point $z = 1$ the point of the order star induced by $z^{-\mu} w_1(z)$ locally consists of $p + 1$ sectors of an angle $\frac{\pi}{p+1}$, separated by $p + 1$ sectors of Ω^c, each with the same angle. □

Observe that Ω usually consist of several disconnected components. We say a component has **multiplicity** m if it contains m of the sectors described in Lemma 4.5.

The basic philosophy of the proof is now as follows. By Lemma 4.4 one can distinguish between the components of Ω inside the unit disk $\Delta = \{(z,w) \in M \,\Big|\, |z| \leq 1\}$ and those outside. At this point one wants to apply the argument principle to $f(z)$. The question arises whether one should work on M or on the noncompact Riemann surface induced by $z^{-\mu}w(z)$. We will work on M and therefore we have to be able to define $z^{-\mu}$ uniquely in the component of Ω on M with which we want to work. On a technical level this is done as follows. M is represented over z as two sheets which are connected through branch cuts. There are only finitely many of these cuts. They connect the branch points. There is not a unique way of choosing these but the connectivity of the Ω components are independent of this choice since it is a property related to M and not to a particular representation over z. Once the cuts have been chosen we call the sheet containing $(1,1)$ the principal sheet and the other one the secondary sheet. To be able to define $z^{-\mu}$ uniquely on M we cut both sheets with two cuts L_1, L_2 which connect $(0,w_i^0)$ with (∞, w_i^∞) such that the projections of the cuts onto the z-plane are identical and do not intersect the cuts used in the representation of M. These cuts L_1, L_2 can always be chosen. We can now define $\log z$ and hence $z^{-\mu} = e^{-\mu \log z}$ uniquely on $M \backslash \{L_1, L_2\}$ and the choice is done such that in a neighborhood of $(1,1) \in M$ $\log z$ is real for z real. When we apply the argument principle on a component we are not allowed to cross either of the cuts L_1 or L_2. In both cases we move along the cut to $z=0$, encircle this point to get to the other side of the cut and move back along the other side of the cut until we reach the same location on the other side. Such integration paths are illustrated in Fig. 2.

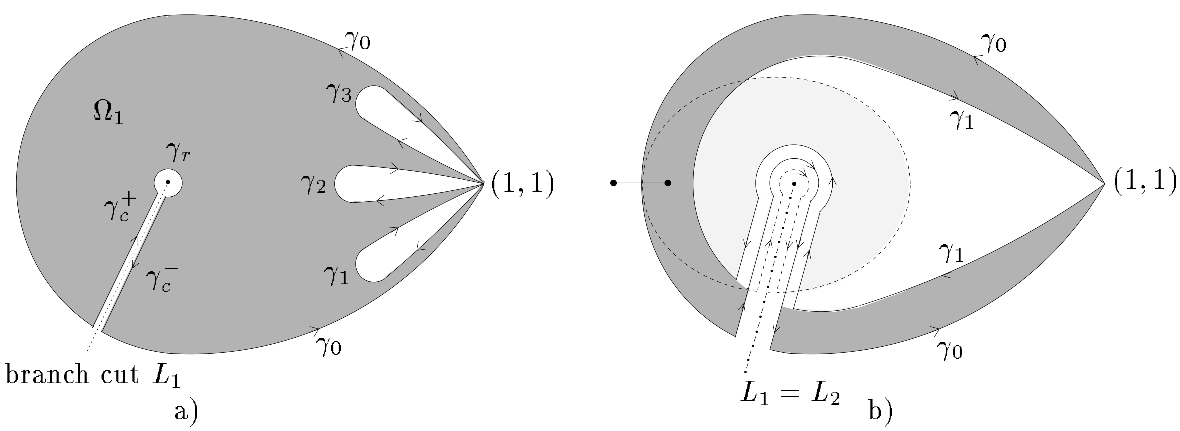

Fig. 2: Ω components Ω_1 with integration paths

Using the argument principle one can show that each component of Ω in Δ has either to contain one of the zero points $(0,w_i^0) \in M$, $i=1,2$; or a pole (z_i, ∞), $i = 1,2,...,r_2$, where $a_2(z_i) = 0$. The order star technique uses the argument principle to relate the order of these poles, i.e. the leading exponent of the expansion of these points, to the multiplicity of the component. This will be done in Section 5 and we shall use this relation to prove the main theorem 3.1 in Section 6.

5 Bounds for the multiplicity of Ω components

We need to look at the expansion of φ at the zeropoints $(0, w_i^0)$ and at the poles of $w(z)$. Let $-\alpha_i$ be the leading exponent of φ at $(0, w_i^0)$.

Conjecture 5.1 The multiplicity m for the Ω-component Ω_1 in Δ containing both zero points $(0, w_i^0)$, $i = 1, 2$, and no poles satisfies

$$m \leq \lfloor\alpha_1\rfloor + 1 + 2\lfloor\alpha_2\rfloor + \max\{1, 2\lfloor\delta_1 + \delta_2\rfloor\},$$

where $\alpha_i = \lfloor\alpha_i\rfloor + \delta_i$, $i = 1, 2$. □

Remark 5.2

The conjecture has been proved in [9] for schemes of optimal order i.e. $p = |I| - 2$ and $-1/2 < \mu < 0$. Due to its length we have to omit this proof here. It works by also counting the number of branch points of the algebraic function. To do this one also has to work with the part of Ω outside Δ.

Note that in general, i.e. if $p < |I| - 2$ or $p = |I| - 2$ and $0 < \mu < 1/2$, we can only prove (19) with a factor 2 in front of the term $\lfloor\alpha_1\rfloor$. In the situation where the two zero points lie in different components we had shown already in [12] the following proposition

Proposition 5.3 *The multiplicity m of an Ω-component Ω_1 in Δ containing exactly one zero point and no poles satisfies either*

$$m \leq \lfloor\alpha_1\rfloor + 1, \ (0, w_1^0) \in \Omega_1 \tag{19}$$

or

$$m \leq 2\lfloor\alpha_2\rfloor + \max\{1, 2\lfloor\delta_1 + \delta_2\rfloor\}, \ (0, w_2^0) \in \Omega_1 \tag{20}$$

where $\alpha_i = \lfloor\alpha_i\rfloor + \delta_i$, $i = 1, 2$. The Ω_1 component which allows bound (19) is called **simple** *while the one allowing bound (20) is called* **binary**. *Two disconnected binary components cannot exist simultaneously.* □

At this point we can indicate where our proving technique to show Conjecture 5.1 fails. Consider the order stars depicted in Fig. 3.

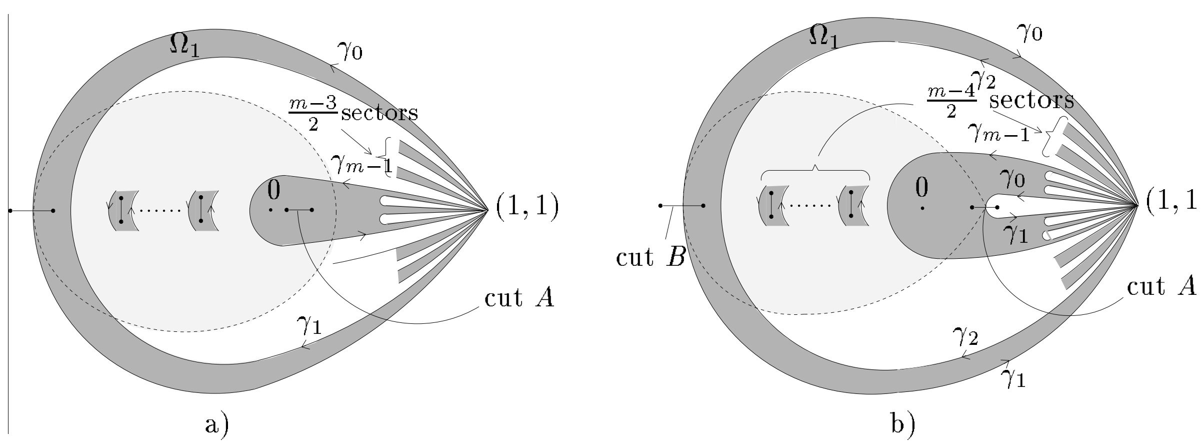

Fig. 3: Example of Ω-component where bound (19) cannot be shown

If in Fig. 3a) the cut A is missing we can apply Proposition 5.3 and get as bound for the total multiplicity the sum of the two bounds (19) and (20) and hence (19) holds. However if the cut A is present our techniques give a factor two in front of $\lfloor\alpha_1\rfloor$. Moreover in Fig. 3b) it is not clear how one could remove the cut A.

In Conjecture 5.1 and Proposition 5.3 the poles of w away from $z = 0$ have been excluded for the following reason. Since we want to bound m for all possible order stars we have to consider those where this bound becomes as large as possible. With similar proofs as the one used to show Proposition 5.3 one sees that the poles give rise to the highest bound when the components of Ω with a zero point do not contain a pole away from zero and each of these components contains exactly one pole and this pole has order 1.

Proposition 5.4 *Assume that φ has q poles away from zero in Ω-components in Δ which don't contain the zero points. Hence the sum m of all the multiplicities of these components satisfies the inequality*

(21) $$m \leq 3q .$$

Equality in (21) occurs if each of these poles is simple and each component contains exactly one of these poles. □

We collect Conjecture 5.1, Proposition 5.3 and 5.4 in the following

Corollary 5.5 *Let m be the total multiplicity of Ω in Δ of a stable normalized scheme with a convex and increasing stencil. Then we have the following three bounds:*

i) If Ω contains no zero point $(0, w_i^0)$ then

(22) $$m \leq 3r_2 .$$

ii) If Ω contains exactly one zero point $(0, w_i^0)$ then

(23) $$m \leq 2\,\lfloor\max\{\alpha_1, \alpha_2\}\rfloor + \max\{1, 2\lfloor\delta_1 + \delta_2\rfloor\} + 3r_2 .$$

iii) If Ω contains both zero points $(0, w_i^0)$ then

(24) $$m \leq \lfloor\min\{\alpha_1, \alpha_2\}\rfloor + 1 + 2\lfloor\max\{\alpha_1, \alpha_2\}\rfloor + \max\{1, 2\lfloor\delta_1 + \delta_2\rfloor\} + 3r_2 .$$

□

Remark 5.6

Observe that Conjecture 5.1 is only used in (24) if $p < |I| - 2$ or if $p = |I| - 2$ and $0 < \mu < 1/2$. Moreover as explained in Remark 4.2 the actual proofs have been carried through for the cases where $z = 0$ is not a branch point of w.

6 Proof of the main theorem

We shall indicate in the proof where we use Conjecture 5.1. Observe, however, that for $p = |I| - 2$ and $-1/2 < \mu < 0$ the conjecture has been proved in [9]. The proof of Theorem 3.1 will be divided into two parts. First we show that

(25) $$p \leq 2R ,$$

where R is the number of downwind points defined in (15). Since the scheme is stable we know that the order star Ω does not intersect the unit circle $\{(z,w) \in M \,\big|\, |z| = 1\}$ and hence all components of Ω in Δ are bounded and the bounds for the multiplicities of m given in Section 5 apply. Since the scheme is normalized we have exactly r_2 poles in Ω in Δ. From Proposition 4.1 we have the following expansions of $\varphi(z)$ at $z = 0$, w_i

$$
\begin{aligned}
\varphi(z) &= z^{-(r_1-r_2)-\mu}(c_0 + c_1 z + c_2 z^2 + \dots) \text{ at } (0, w_1^0) \\
\varphi(z) &= z^{-(r_0-r_1)-\mu}(d_0 + d_1 z + d_2 z^2 + \dots) \text{ at } (0, w_2^0) .
\end{aligned} \tag{26}
$$

It should be noted here that we have no means of associating a certain expansion with the zero point on a specific sheet. Hence the expansions will be associated with the zero points in the way which leads to the highest possible multiplicity.

In the remainder of the proof we have to work separately with the cases where $\mu < 0$ and where $\mu > 0$.

a) We first assume $-\frac{1}{2} < \mu < 0$. Then the following choices of the indices r_0, r_1, r_2 lead to different combinations of Ω-components inside Δ.

  (i) $r_0 = r_1 = r_2 = 0$. Then also $R = 0$. According to the Courant-Friedrichs-Lewy condition the scheme cannot be convergent, i.e. it is impossible to have simultaneously order $p \geq 1$ and stability.

  (ii) $r_0 = r_1 = r_2 > 0$. There are r_2 poles away from $z = 0$ inside Δ, while $\varphi(z)$ has positive leading exponents of $-\alpha_i = -\mu$ at $z = 0$ on both sheets of M, implying that both zero points belong to Ω^c. We apply Proposition 5.4 or i) of Corollary 5.5 to obtain

$$m \leq 3r_2 = r_0 + r_1 + r_2 = R .$$

  (iii) $0 = r_0 - r_1 < r_1 - r_2$. Then $-\alpha_1 = -\mu > 0$, implying that $(0, w_1^0) \in \Omega^c$, and $-\alpha_2 = -(r_1 - r_2) - \mu < 0$, implying that $(0, w_2^0) \in \Omega$. We apply ii) of Corollary 5.5. Then we have

$$
\begin{aligned}
m &\leq 3r_2 + 2\lfloor r_1 - r_2 + \mu \rfloor + 2\lfloor 2 + 2\mu \rfloor \\
&= 3r_2 + 2(r_1 - r_2 - 1) + 2 \\
&= r_2 + 2r_1 = r_2 + r_1 + r_0 = R .
\end{aligned}
$$

  (iv) $0 < r_0 - r_1 < r_1 - r_2$. Then $-\alpha_1 = -(r_0 - r_1) - \mu < 0$ and $-\alpha_2 = -(r_1 - r_2) - \mu < 0$, implying that both $(0, w_1^0)$ and $(0, w_2^0)$ belong to Ω. We apply iii) of Corollary 5.5. Then the total multiplicity m inside Δ is bounded by

$$
\begin{aligned}
m &\leq 3r_2 + \{2\lfloor r_1 - r_2 + \mu \rfloor + 2\lfloor 2 + 2\mu \rfloor\} + \{\lfloor r_0 - r_1 + \mu \rfloor + 1\} \\
&= 3r_2 + \{2(r_1 - r_2 - 1) + 2\} + \{(r_0 - r_1 - 1) + 1\} \\
&= r_0 + r_1 + r_2 = R .
\end{aligned}
$$

b) Assume $0 < \mu < \frac{1}{2}$. Then we have $-\alpha_1 = -(r_0 - r_1) - \mu < 0$ and $-\alpha_2 = -(r_1 - r_2) - \mu < 0$, implying that both $(0, w_1^0)$ and $(0, w_2^0)$ belong to Ω. We apply iii) of Corollary 5.5. This leads to the bound

$$
\begin{aligned}
m &\leq 3r_2 + \{2\lfloor r_1 - r_2 + \mu \rfloor + 1\} + \{\lfloor r_0 - r_1 + \mu \rfloor + 1\} \\
&= 3r_2 + \{2(r_1 - r_2) + 1\} + \{(r_0 - r_1) + 1\} \\
&= r_0 + r_1 + r_2 + 2 = R .
\end{aligned}
$$

Observe that in the case b) and iv) of a) we have used Conjecture 5.1 if $p < |I| - 2$ and if $p = |I| - 2$ and $0 < \mu < 1/2$. Moreover in both cases it could happen that w has a branch point at $z = 0$, see Remark 4.2.

In all cases we obtain

$$m \leq R .$$

The remainder of the proof makes use of Lemma 4.5. Observe that by this lemma and stability the multiplicity outside Δ can be at most $m + 1$. Hence we find

$$p + 1 \leq m + (m + 1) \leq 2R + 1$$

which leads to the bound (25) $p \leq 2R$.

Concerning the bound $p \leq 2S$ we could work in exactly the same fashion but outside Δ. However by using the mapping $z \rightarrow 1/z$ and $\mu \rightarrow -\mu$ we can apply the bound just shown. To do this our assumptions on the scheme and the bound just proved have to be invariant with respect to this mapping. This completes the proof of Theorem 3.1. □

Remark 6.1

It is interesting to observe the following since the order condition represents a regular system of linear equations. The presence of a stencil point raises the highest reachable order by 1. To each stencil point $(2, j)$ on the newest time-level belong, due to our assumption of an increasing stencil, two stencil points $(1, j)$ and $(0, j)$, which means by collecting these that for our schemes a stencil point $(2, j)$ raises the highest reachable order by 3. This factor can be found in all three bounds of Corollary 5.5 associated with r_2. In a certain sense we can say the stencil points $(0, j)$, $(1, j)$ with $j \in \{-r_2, -r_2 + 1, ..., -1\}$ are used up by the implicit stencil points $(2, j)$, $j \in \{-r_2, -r_2 + 1, ..., -1\}$. For simplicity assume that (16) is satisfied, i.e. $0 \leq r_0 - r_1 < r_1 - r_2$. Hence $\alpha_2 = r_1 - r_2 + \mu > \alpha_1 = r_0 - r_1 + \mu$. Again to each stencil point $(1, j)$ which has not been used by the implicit part, i.e. $j \in \{-r_1, -r_1 + 1, ..., -r_2 - 1\}$ we can associate one stencil point $(0, j)$. Again collecting these one has that each $(1, j)$, $j \in (-r_1, ..., -r_2 - 1\}$ raises the highest reachable order by 2. This factor can be found in the bounds (23) and (24) in front of the term with the larger α_i which according to our assumptions is the α_i associated with the stencil points on the time level t_{n+1} which have not been used by the implicit part. Similarly one sees that the stencil points $(0, j)$ which have not been used up by the other time levels i.e., the ones with $j \in \{-r_0, -r_0 + 1, ..., -r_1 - 1\}$ are associated with the factor 1 in front of smaller of the two α_i in bound (24). In further investigations to more than 3 time level schemes we have already discovered that this pattern continues.

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