Absorbing boundary conditions for acoustic and elastic waves

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Introduction

In this paper I review the use of approximate one-way wave equations (OWWES) obtained from rational approximations to $\sqrt{1-s^2}$ as absorbing boundary conditions for acoustic and elastic waves. In particular, I will present a procedure which allows for very easy implementation of high-order boundary conditions for acoustic waves. Similar techniques applied to the elastic wave equation confirm that an Ansatz of Engquist and Majda [2] about the form of a low order boundary condition is correct.

Acoustic waves.

As the model for acoustic wave propagation consider the solution $u = u(x, y, t)$ of the second order wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}) \tag{1}$$

for $x > 0$, $y \in \mathbf{R}$, $t > 0$. The dispersion relation for the plane waves, $u(x, y, t) = e^{i(\omega t+\xi x+\eta y)}$ which satisfy (2.1) is given by

$$\omega^2 = c^2(\xi^2 + \eta^2) \tag{2}$$

and their phase velocity by $c(-c\xi/\omega, -c\eta/\omega)$. If an artificial boundary is placed at $x = a$ it should be absorbing and hence allow only those waves for which $-\xi/\omega > 0$ to be propagated. This may be achieved by choosing the negative square root in the relationship

$$\xi = \pm\frac{\omega}{c}\sqrt{1-\left(\frac{\eta c}{\omega}\right)^2} \tag{3}$$

so that ξ and ω are of opposite sign. In order to convert (2.3) back into a form useful for implementing as a boundary condition at the $x = a$ boundary approximations to the function $f(s) = \sqrt{1 - s^2}$, $s = \eta c/\omega$, are required.

Lindman [7] was the first to propose the use of approximate OWWEs as absorbing boundary conditions. His method is derived from a degree [6|6] rational approximation, where here the notation $[m|n]$ is a rational approximation of degree m in the numerator and n in the denominator, to $1/\sqrt{1 - s^2}$. Despite the effectiveness of this boundary condition other boundary conditions derived later [1, 2], the so-called "paraxial" approximations, have gained wider acceptance in the literature. The most widely used paraxial approximation is the [2|0] approximation which is very good for absorbing waves near normal incidence but does not absorb waves of glancing incidence. Thus, in comparison to Lindman's boundary condition it is far less effective.

To overcome this problem Halpern and Trefethen [3] investigated several classes of approximate OWWEs. Included in these are least squares approximations and minimax approximations, both of which seek to minimize the reflection for all incident angles. All of these approximations can be implemented using an approach suggested by Lindman, described in [9].

Suppose that the $[m|n]$ rational approximation $r(s) = P_m(s)/Q_n(s)$ to $\sqrt{1 - s^2}$ has been found by one of the procedures in [3]. In order to use the formulation in [7], this approximation is rewritten in the form

$$r(s) = p_0\left(1 + \sum_{i=1}^{N} \frac{\alpha_i s^2}{1 - \beta_i s^2}\right) \tag{4}$$

where $2N = \max\{m, n\}$. Substitution of (2.4) into (2.3) with the negative sign gives the system of equations

$$p_0 u_t + c u_x = -p_0 \sum_{i=1}^{N} h_i$$

where

$$\frac{\partial^2 h_i}{\partial t^2} - \beta_i c^2 \frac{\partial^2 h_i}{\partial y^2} = c^2 \alpha_i \frac{\partial^3 u}{\partial y^2 \partial t}, \qquad i = 1 : N \tag{5}$$

to be solved at the boundary. These equations are not identical to those used in [7] where the approximation $R(s)$ to $1/\sqrt{1 - s^2}$ is used. The difference approximation, see [9], to these equations is thus also a modification of that proposed in [7]. It is shown in [10], however, that in situations where $r(s) = 1/R(s)$, the stability properties of the numerical schemes are identical. Renaut [9] has proved necessary conditions for stability of the $[m|n]$ schemes with $(m, n) \in \{(2, 0), (2, 2), (4, 2)\}$. For none of the approximations given in [3] do these conditions impose restrictions on the Courant number, $\Delta t/\Delta x$, less than that imposed by the stability of the interior central difference approximation.

Observe that (2.5) can now be easily implemented for any order absorbing boundary condition. Increasing the order amounts to changing the coefficients,

α_i, β_i, and introducing new functions h_i. Furthermore, defining $k = (m+n+2)/2$, an OWWE with k even, $k = 2l$, requires the same amount of work as one with k odd, $k = 2l+1$. Therefore it is more effective to use only those schemes for which k is odd.

I now consider the question of whether there is some advantage to be gained by using the rational approximations to $1/\sqrt{1-s^2}$ instead of $\sqrt{1-s^2}$. The situations $r(s) = 1/R(s)$ arise only in the cases where $r(s)$ and $R(s)$ are both derived from some interpolating conditions to $\sqrt{1-s^2}$ or $1/\sqrt{1-s^2}$, respectively (see [10]). Least squares and minimax approximations to $1/\sqrt{1-s^2}$, on an interval approaching $s = \pm 1$, to avoid the discontinuity at $s = \pm 1$ are presented in [8]. They do not show reflection coefficients which are obviously superior to those for the $\sqrt{1-s^2}$ case. But numerical tests indicate no instabilities which are seen for the $r(s)$ approximations due to the existence of generalized eigensolutions. In Figure 1 some comparisons which show the effectiveness of these boundary conditions are presented (for further details, see [8]).

Elastic waves.

Consider the model of elastic wave propagation for a nonviscous isotropic medium given by

$$\begin{aligned} \begin{pmatrix} u \\ w \end{pmatrix}_{tt} &= \begin{pmatrix} c_p^2 & 0 \\ 0 & c_s^2 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}_{xx} + \begin{pmatrix} c_s^2 & 0 \\ 0 & c_p^2 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}_{yy} \\ &+ \begin{pmatrix} 0 & c_p^2 - c_s^2 \\ c_p^2 - c_s^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}_{xy}. \end{aligned} \tag{6}$$

Here $U = (u, w)^T$ where $u = u(x, y, t)$ and $w = w(x, y, t)$ denote the horizontal and vertical displacement fields. c_p and c_s are velocities of the compressional, P-waves and shear waves, S-waves, respectively, which are determined by the physical parameters of the medium. Typically c_p/c_s around $\sqrt{3}$ is observed but much larger values as high as 2.5 or 3.0 are encountered in less rigid materials.

Following the derivation in [2] nonlocal perfectly radiating boundary conditions for oscillatory waves at $x = a$ are given by

$$\tilde{q}_p^T \cdot \left(\frac{dU}{dx} - ik_p(l, \omega) U \right) = 0$$

and

$$\tilde{q}_s^T \cdot \left(\frac{dU}{dx} - ik_s(l, \omega) U \right) = 0. \tag{7}$$

Here the wavenumbers k_p and k_s are given by $\omega^2 = c_p^2(k^2 + l^2)$ and $\omega^2 = c_s^2(k^2 + l^2)$ and correspond to dispersion relations for P-waves and S-waves, respectively, see [2]. The eigenvectors associated with these two types of waves are given by $q_p = (k, l)^T$ and $q_s = (-l, k)^T$, respectively. The vectors $\tilde{q}_p$ and $\tilde{q}_s$ are then defined by the requirement $\tilde{q}_i^T \cdot q_j = 0$, $i \neq j$, $i, j = s$ or p. Since the phase velocity is given by $c((ck/\omega),(cl/\omega))$, $c = c_p$ or $c = c_s$, waves for which k and ω are of the same sign are required to give absorption at $x = a$. Hence, $k_i = \frac{\omega}{c_i}\sqrt{1-(c_i l/\omega)^2}$, $i = p, s$ and we observe that again approximations to $\sqrt{1-s^2}$ are of interest.

At this point I depart from the derivation in [2] and continue to use k_p and k_s without approximation. Using $\tilde{q}_p = (k_s, l)^T$ and $\tilde{q}_s = (-l, k_p)^T$ in equations (3.2) gives

$$\begin{aligned} k_s\frac{du}{dx} + l\frac{dw}{dx} - ik_p k_s u - ilk_p w &= 0 \\ -l\frac{du}{dx} + k_p\frac{dw}{dx} + ik_s iu - ik_s k_p w &= 0. \end{aligned} \tag{8}$$

Engquist and Majda [2] observed that the $\partial^2/\partial x\partial y$ terms in this expression tend to give unstable difference approximations. Here I seek to eliminate these terms by solving for du/dx and dw/dx in (3.3) to give

$$\begin{aligned} \frac{du}{dx}(k_p k_s + l^2) - uik_s\frac{\omega^2}{c_p^2} + ilw(k_p k_s - k_p^2) &= 0 \qquad (9) \\ \frac{dw}{dx}(k_p k_s + l^2) - wik_p\frac{\omega^2}{c_s^2} + ilu(k_s^2 - k_p k_s) &= 0 \end{aligned}$$

As an attempt to find a simple implementation of these equations which would allow easy adaptation to higher order I proceed as in the acoustic case by using the Lindman form of approximations to $\sqrt{1-s^2}$ and use first order approximations to the terms $(k_p k_s + l^2)$ and $(k_p k_s - k_p^2)$. In this way the equations

$$\begin{aligned} \frac{\gamma}{2c_s}\frac{du}{dx} + p_0\frac{du}{dt} + p_0^2(c_p - c_s)\frac{dw}{dy} &= -p_0\sum_{i=1}^{N} h_i \\ \frac{\gamma}{2c_p}\frac{dw}{dx} + p_0\frac{dw}{dt} + p_0^2(c_p - c_s)\frac{du}{dy} &= -p_0\sum_{i=1}^{N} g_i \end{aligned}$$

where

$$\begin{aligned} \frac{\partial^2 h_i}{\partial t^2} - \beta_i c_s^2\frac{\partial^2 h_i}{\partial y^2} &= \alpha_i c_s^2\frac{\partial^3 u}{\partial^2 y\partial t} \\ \frac{\partial^2 g_i}{\partial t^2} - \beta_i c_p^2\frac{\partial^2 g_i}{\partial y^2} &= \alpha_i c_p^2\frac{\partial^3 w}{\partial^2 y\partial t} \end{aligned} \tag{10}$$

result. Here $\gamma = c_s^2 + c_p^2 - p_0^2(c_s^2 + c_p^2 - 2c_s c_p)$. For the second order paraxial approximation, $p_0 = 1$, $p_2 = -1/2$, the absorbing boundary condition

$$\left(\frac{\partial^2}{\partial x\partial t} + A_0\frac{\partial^2}{\partial t} + A_1\frac{\partial^2}{\partial y\partial t} + A_2\frac{\partial^2}{\partial y^2}\right)U = 0 \tag{11}$$

at $x = a$ is obtained. The matrices are

$$A_0 = \begin{pmatrix} \frac{1}{c_p} & 0 \\ 0 & \frac{1}{c_s} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \frac{C_p - C_s}{C_p} \\ \frac{C_p - C_s}{C_s} & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -\frac{1}{2}\frac{C_s^2}{C_p} & 0 \\ 0 & -\frac{1}{2}\frac{C_p^2}{C_s} \end{pmatrix}.$$

Observe that (3.7) has the same form with a different A_2, as the Ansatz in [2] but is obtained without any assumptions other than those on the order of the approximations to k_s and k_p. The difference approximations for the equations in (3.5) are obtained analogously to those in (2.5) with the u_y term approximated by $\frac{D_0^y}{2}(U_{Mk}^n + U_{Mk}^{n+1})$ where $U_{mk}^n \approx u(m\Delta x, k\Delta x, n\Delta t)$, $M\Delta x = a$.

The numerical tests indicate that (3.6) is better than that suggested in [7] and compares favorably with the schemes presented by Higdon [6]. The system (3.5) is very easily implemented for higher order approximations to k_p and k_s provided that the $(k_pk_s + l^2)$ and $(k_pk_s - k_p^2)$ terms remain first order. Numerical experimentation with P-waves confirms that (3.5) is useful for improving the order of accuracy of absorbing boundary conditions. But when the initial condition is an S-wave (3.5) is not satisfactory. Examples of these comparisons are given in Figures 2 and 3.

I can very easily explain why (3.5) does not work for S-waves. An incident S-wave, incident at angle θ, on a boundary may give rise to either reflected S or P-waves. For $0 \leq \theta \leq \sin^{-1}(c_p/c_s)$ reflected P-waves are oscillatory but when $\theta > \sin^{-1}(c_p/c_s)$ these waves are evanescent, $c_pl/\omega > 1$. Therefore, k_p must be replaced by ik_p and approximations to $\sqrt{1-s^2}$ for $s \in (-1,1)$ will not absorb reflected evanescent P-waves. Hence the problem with (3.5) may be resolved by searching for numerically useful approximations to k_p or ik_p depending on s (see for example [7]).

Conclusion.

In this paper I have presented a technique which allows for easy implementation of absorbing boundary conditions of any order for acoustic waves. The ideas can be extended to elastic waves but require the correct form of the approximations to allow for evanescent waves. The latter is work in progress and will be reported in a future report along with more details of the stability issues in the elastic wave situation. The methods for acoustic waves have been demonstrated to be easier to implement at high order than those given by Higdon [4, 5] and to be more effective than low order equations.

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