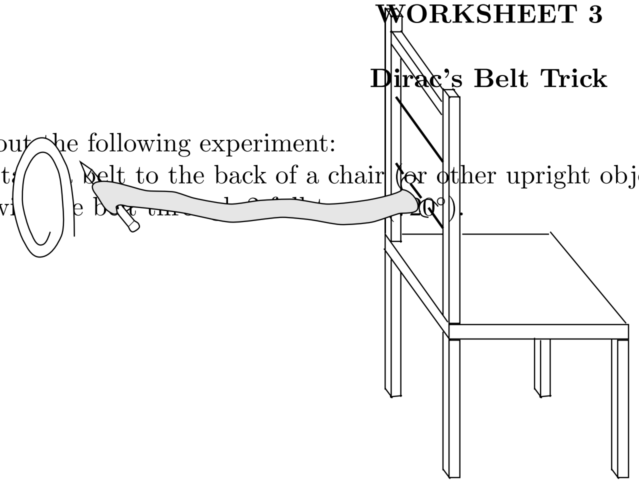
Emerging Scholars
University of Texas

Spring 1998

WORKSHEET 3

Dirac's Belt Trick

1. Carry out the following experiment:
   Attach a belt to the back of a chair (or other upright object).
   Twist the belt through 2 full turns (720°).



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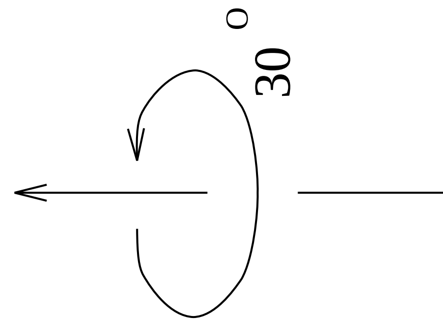
2. Try the same experiment with more or less than 2 full turns.

3. The mathematical object that explains the above behavior is called

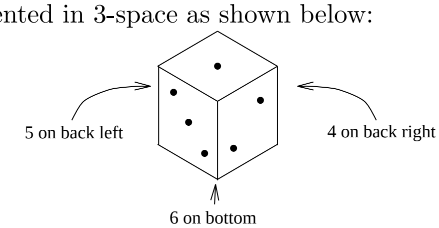
   the space of all rotations in 3-dimensions.

   We now try to picture this object.

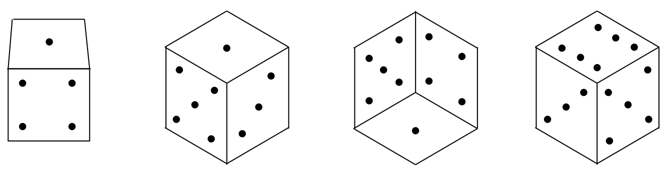
   A rotation in 3-dimensions is determined by two pieces of information: the axis about which to rotate and an angle through which we rotate. For example, rotate 30° about the vertical axis. (We need to pick a "sense" through which to rotate - here, we choose a "right handed" rule.)



   a) Consider a single die oriented in 3-space as shown below:



   In the following four pictures the die has been rotated about some axis by some angle. For each picture, draw the axis of rotation through the die and indicate the angle of rotation.



   What you have done is identify exactly which rotation in "the space of all rotations in 3-dimensions" moved the die from the orientation in the original picture to each of those above.

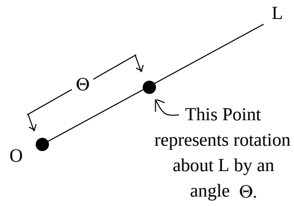
   b) Draw a die which has been moved from its original orientation as given in part a) by each of the following rotations:

   i) angle: $\pi/2$ axis: to the right, in the plane of the page
   ii) angle: $\pi/2$ axis: to the left, in the plane of the page
   iii) angle: π axis: coming straight **out** of the plane of the page
   iv) angle: π axis: going straight **into** the plane of the page

   Are rotations iii) and iv) really different?

4. Now we construct an actual **picture** of "the space of all rotations in 3-dimensions":

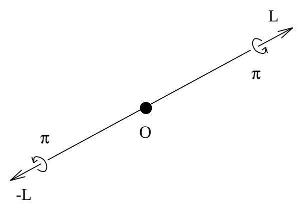
   Consider a solid ball of radius π. Think of the center point as "zero rotation" or the rotation of 3-space by **zero** angle (through any axis). Now, any other rotation will also be represented by a point in this solid ball. Specifically, *a rotation through an angle θ about an axis L will be represented by a point a distance θ out from the center along the axis L.*



   a) Locate a point in the solid ball that corresponds to each of the eight rotations from problem 3.

   b) Do some of these rotations correspond to more than one point in the ball?

5. Now notice that a rotation through π (180°) about L is the same as a rotation through π about the axis pointing in the opposite direction to L:



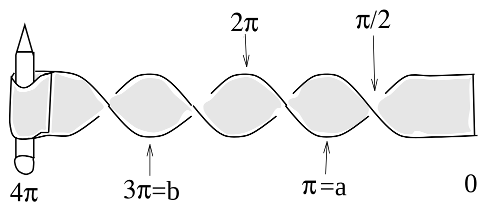
   Therefore in our picture, we need to consider a point on the surface of the ball (at distance π from the center) along an axis L as the **same** rotation which is represented by the point on the opposite side of the surface (the antipodal point).

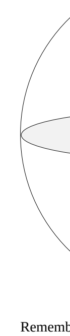
   We are led to conclude that "the space of all rotations in 3-dimensions" can be pictured as a solid ball with antipodal points on the boundary identified as the same rotation. Every rotation is represented by a point on this ball, and every point represents a rotation.

   Suppose you are a creature who lives inside of this ball. If you take a walk from the center (the zero rotation) to a point on the boundary (a rotation by angle π) notice that you can keep walking! You will just reappear on the opposite side of the ball coming inwards since we have identified these points! (Just like a 3-d video game....) The question is this: if you do this and then return straight to the center, you will notice something strange. What?

6. With the above picture in mind, we can now explain the Dirac belt:

   When we have rotated the belt through 2 turns, the belt contains a graphical illustration of a path through our picture. The path makes two "laps" through the space: beginning at the center, travelling out to point "a," back to the center, out to point "b" then back to the center again.





   The sequence of moves which dissolves the rotations while keeping 0 and 4π fixed is shown below. Here, we only show the disk where all of the action is: