Emerging Scholars
University of Texas

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WORKSHEET 27

1. The rate at which any radioactive material decays is directly proportional to the amount of the material present. That is,

$$\frac{dS}{dt} = -kS$$

where $S(t)$ is the amount present at time t and k is a positive constant. Suppose $k = 1$, there is initially 1 kg of radioactive material present, and t is measured in days.

   a) Over the first 12 hours, approximate the rate of decay by a constant (even though the rate is changing). At that rate, how much material would be left at the end of 12 hours?

   b) Now over the next 12 hours approximate the rate of decay by another constant. At this rate of decay, how much material would be left at the end of the day?

   c) Now start over at the beginning of the day, dividing it up into 24 one hour periods. Starting with hour one, approximate the rate of decay over that hour and find the amount of material that would remain at the end of that hour. Repeat this until you have an estimate for how much material is left at the end of the day.

   d) If you had divided the day up into minute intervals in order to improve accuracy, would your new estimate be bigger or smaller than that in part c?

   e) Use your calculator to find $1/e$.

2. a) For what values of b is $f(x) = \log_b x$ an increasing function?

   b) Sketch the functions $\log_e x$ and $\log_{1/e} x$.

   c) For what values of b is $f(x) = b^x$ an increasing function?

   d) Sketch the functions e^x and $(1/e)^x$.

3. As a promotional gimic, your bank offers you the option of investing a single dollar at 100% interest for the year, that is, they will pay you back two dollars at the end of the year. The bank down the street offers to compound the interest every month. (What does this mean?) A third bank offers to compund the interest hourly!

   a) How much will be paid out by the second bank at the end of one year?

   b) How much will be paid out by the third bank at the end of one year?

   c) What is meant by continuously compounded interest?

4. Explain how the constant e might appear in a model of population growth.

5. Find an estimate for the infinite sum

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots$$

6. a) Sketch the graph of a nonzero function which is its own derivative, that is, its slope is equal to its height at every point.

   b) Suppose that such a function has $f(0) = 1$. What is the slope of the tangent line at $x = 0$? What is the height of this tangent line at $x = 1.1$? Why might this be used as an approximation for $f(1.1)$? Repeat this process nine more times to find an estimate for $f(1)$. Is this an overestimate or an underestimate?