Emerging Scholars
University of Texas

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WORKSHEET 23

Note: In this worksheet, you will be expected to work with *Riemann sums* (sometimes called *summation notation*). If $a_1, a_2, a_3, \ldots, a_i, \ldots, a_{n-1}, a_n$ are n numbers, then the meaning of the Riemann sum $\sum_{i=1}^{n} a_i$ is

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots + a_{n-1} + a_n.$$

1. Consider the following definitions:

   Definition 1. Let $a = x_0 \le x_1 \le x_2 \le \cdots \le x_n = b$ be a partition of the interval $[a, b]$. The *mesh* of the partition is the width of the largest subinterval $[x_{i-1}, x_i]$.

   Definition 2. If f is a function defined on $[a, b]$ and the sums $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$ approach a certain number as the mesh of partitions of $[a, b]$ shrinks toward 0 (no matter how the sampling numbers c_i are chosen in $[x_{i-1}, x_i]$), that certain number is called *the definite integral of f from a to b*. It is denoted

   $$\int_a^b f(x)\,dx.$$

   In short, the definite integral of f over $[a, b]$ is

   $$\lim_{\text{mesh}\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

   a) Draw a picture of a sequence of partitions in which the mesh does not go to 0.

   b) Explain why a sequence of partitions such as the ones you drew in part a) are not be allowed in Definition 2.

   c) Sketch two identical copies of the graph of some function f on an interval $[a, b]$. Partition the interval in both pictures, using a finer partition for the second graph. Now, draw in rectangles which explain the definition of the definite integral.

2. Write the following limits as definite integrals:

   $$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n} \qquad \lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{2i}{n}\right)^2 \frac{2}{n}$$

3. Write the following definite integrals as limits:

   $$\int_0^1 x + 2\,dx \qquad \int_{-3}^{3} x^2 - x\,dx$$

4. Use the formulas

   $$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

   to evaluate the Riemann sums you wrote down in Problem 3. Then compute the limits as $n \to \infty$.

5. Use the Second Fundamental Theorem of Calculus to check your answers in Problem 4.