

WORKSHEET 12

- Imagine a road on which the speed limit is specified at every single point. In other words, there is a certain function L such that the speed limit x miles from the beginning of the road is $L(x)$. Two cars A and B, are driving along this road; car A's position at time t is $a(t)$, and car B's is $b(t)$.
 - What equation expresses the fact that the car A always travels at the speed limit?
(Hint: the answer is *not* $a'(t) = L(t)$.)
 - Suppose that A always goes at the speed limit, and that B's position at time t is A's position at time $t - 1$. Show that B is also going at the speed limit at all times.
 - Suppose B always stays at a constant distance behind A. Under what circumstances will B still always travel at the speed limit?

- Prove that it is impossible to find two differentiable functions $f(x)$ and $g(x)$ for which $f(0) = g(0) = 0$ and which satisfy $f(x)g(x) = x$ for all x .
(Hint: differentiate.)

- Assume that $f(x)$ is a differentiable function and that the values of $f(x)$ and its derivative at the points $x = 0, 1, 2$, and 3 are given by:

$$\begin{array}{cccc} f(0) = 3 & f(1) = 5 & f(2) = -2 & f(3) = 6 \\ f'(0) = -1 & f'(1) = 0 & f'(2) = 3 & f'(3) = 1. \end{array}$$

Let $g(x) = x^2 - 3x + 2$. For each function below, calculate the derivative at the given point.

- $f(x) + g(x); \quad x = 0$
- $\frac{f(x)}{g(x)}; \quad x = 1$
- $f(x)g(x); \quad x = 2$
- $\frac{f(x)g(x)}{f(x) + g(x)}; \quad x = 3$
- $f(g(x)); \quad x = 0$
- $f(g(x)); \quad x = 1$
- $g(f(x)); \quad x = 2$
- $g(f(x)); \quad x = 3$

- Prove that the formula for the derivative for an inverse function is

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

(Hint: Let $g(x) = f^{-1}(x)$, then $f(g(x)) = x$. Differentiate.)

- Find $f^{-1}(x)$ given that $f(x) = \frac{2x - 3}{x + 2}$.
 - Differentiate $f^{-1}(x)$ from part b) and compare with the derivative you get by applying the formula in part a).
- Let $f(x) = x^3 + 3x^2 + 3x + 5$. Does f have an inverse? How do you know? If f has an inverse, determine $f^{-1}(5)$, $(f^{-1})'(5)$, $f^{-1}(12)$, and $(f^{-1})'(12)$.

6. When you see the function $x \mapsto \sin x$ it is assumed that x is given in radians. Define a **different** function f to be the function such that $f(x)$ is the value of the sine of an angle which measures x **degrees**. (NOTE: These are indeed different functions! For most values of x , $f(x) \neq \sin x$. Why not?) Let g be defined similarly for cosine.
- Express f and g in terms of \sin and \cos .
 - What is $\frac{df}{dx}$? What is $\frac{dg}{dx}$? (Hint: Use part a) and the chain rule.)
 - Express $\frac{df}{dx}$ and $\frac{dg}{dx}$ in terms of $f(x)$ and $g(x)$. (No mention of \sin or \cos allowed.)
 - Is it still true that $(f(x))^2 + (g(x))^2 = 1$?
 - Why don't we use the unit of degrees in calculus?
7. Find $\frac{dy}{dt}$ given that $y = \frac{3x^3}{2x^2 + 1}$ and that $x = \tan(t^2)$.
8. For this problem, you may want to review your result for the derivative of an inverse from Problem 4.
- What is $(\arccos(x))'$? Write an expression which is free of trig functions. (Hint: Use the formula in part a. To help you simplify, draw a triangle.)
 - Let $y = \tan^2(\arccos(x))$. Find $\frac{dy}{dx}$ using the chain rule.
 - Find another expression for y which doesn't mention trig functions. (Hint: Again, draw a triangle.)
 - Find $\frac{dy}{dx}$ from your expression in part d). How does it compare with your answer in part a)?