WORKSHEET 38

1. Recall that when integrating a function over a region in the plane using polar coordinates, a "magical" factor of r appears in the integrand. What about other coordinate systems? In general, we must include some factor which makes up for the stretching and shrinking that occurs in such a transformation.

   a) For the following transformation, draw the region of the xy-plane corresponding to $0 \le u, v \le 1$:
   $$x = u + 2v \qquad y = -u.$$

   b) What is the area in the xy-plane of a region for which
   $$u_0 \le u \le u_0 + \Delta u \qquad \text{and} \qquad v_0 \le v \le v_0 + \Delta v?$$

   c) Does this depend on where in the plane this takes place (i.e., does it depend on u_0 and v_0)? Guess what "correction factor" must be included when converting to an integral in the uv-coordinate system.

2. a) Sketch the set of all points (r, θ) in polar coordinates on the plane that satisfy the inequalities
   $$r_0 \le r \le r_0 + \Delta r \qquad \text{and} \qquad \theta_0 \le \theta \le \theta_0 + \Delta\theta.$$

   b) Find the area between the circles of radii r_0 and $r_0 + \Delta r$.

   c) What fraction of the area in b) falls between two rays at angles differing by $\Delta\theta$?

   d) Show that the area of the region in part a) is precisely
   $$\left(r_0 + \frac{\Delta r}{2}\right) \Delta r \Delta\theta.$$

   e) Use the result from part d) to justify the correction factor r that appears when integrating in polar coordinates.

3. In general, we may wish to convert an integral from Euclidean (x, y) coordinates to some other coordinate system (u, v). Note that x and y can be computed in terms of u and v, that is, $x = x(u, v)$ and $y = y(u, v)$. Then the correction factor which we must include in the integral for the (u, v)-coordinate system is given by
   $$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

   Here, J is called the *Jacobian* matrix. The vertical bars indicate to take the *determinant* of J, which for the 2-dimensional case, is what is written out on the right-hand side above.

   a) Compute $|J|$ for the coordinate system in Problem 1. Was your guess correct?

   b) Compute $|J|$ for the polar coordinate system.

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4. a) Sketch the region in the xy-plane corresponding to $0 \leq a, t \leq 1$ where (a, t) is the coordinate system given by

$$x = 4a\cos 2\pi t \qquad y = a\sin 2\pi t.$$

(Hint: For a fixed value of a, what object is described by these equations?)

b) In what circumstances might these coordinates be useful?

c) Compute $|J|$ for this coordinate system.

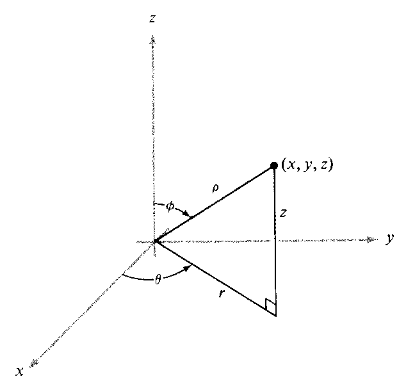
5. Convert the function $f(x, y) = x$ into (u, v)-coordinates (from Problem 1) and into (a, t)-coordinates (from Problem 4). Compute the integral of this function over the regions you sketched in Problems 1 and 4 using BOTH the new coordinates and Euclidean coordinates.

6. To take advantage of spherical symmetries (which appear when modelling many physical situations) spherical coordinates are often used. Let (ρ, θ, ϕ) be coordinates on three-space as follows:

ρ - distance from the origin;

θ - polar angle (measured in the xy-plane);

ϕ - zenith angle (measured from the z-axis).



a) Determine x, y, and z as functions of ρ, θ, and ϕ.

b) Compute the 3×3 determinant $|J|$ for spherical coordinates.

c) Use your answer to part b) to prove that the volume of a ball of radius R is $4/3\pi R^3$.

7. Recal that the area bounded by the polar graph $r = f(\theta)$ between the rays $\theta = a$ and $\theta = b$ is given by the definite integral

$$\int_a^b \frac{1}{2}(f(\theta))^2\, d\theta.$$

Prove this using the double integral in polar coordinates.

How did we prove this earlier in the semester?