WORKSHEET 33

1. a) State Newton's second law of motion (the one relating force and acceleration) and Newton's law of gravity.

   b) State Kepler's three laws of planetary motion.

2. Let $\mathbf{R}$ be the position vector of a planet moving about a stationary sun at the origin. Show that $\mathbf{R}$ stays within a fixed plane.

   Hint: consider $\dfrac{d}{dt}(\mathbf{R} \times \mathbf{R}')$.

3. Because of Problem 2, we can now work in a plane. Let's establish a coordinate system so that this is the x, y-plane (or r, θ-plane), and the planet is somewhere on the x-axis (or $\theta = 0$ ray) at time $t = 0$. Define two pairs of unit vectors

$$\mathbf{u}_r = (\cos\theta, \sin\theta) \quad \mathbf{u}_\theta = (-\sin\theta, \cos\theta) \qquad\qquad \mathbf{i} = (1, 0) \quad \mathbf{j} = (0, 1)$$

   a) Write the position vector $\mathbf{R}$ in terms of $\mathbf{i}$ and $\mathbf{j}$, then in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$.

   b) Write the velocity vector $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$, then in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$.

   c) Write the acceleration vector $\mathbf{a}$ in terms of $\mathbf{i}$ and $\mathbf{j}$ then, in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$.

4. Give the formula for Newton's law of gravity in terms of the vecors $\mathbf{i}$ and $\mathbf{j}$, then give the formula in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$.

5. Using whichever is more convenient, the $\mathbf{i}, \mathbf{j}$-system or the $\mathbf{u}_r, \mathbf{u}_\theta$-system, write down the differential equations governing planetary motion by combining Newton's laws and the formula for acceleration from Problem 3c).

6. Show that the area swept out by the vector $\mathbf{R}$ from time 0 to time t is

$$A(t) = \int_0^t \frac{1}{2} r^2 \theta' \, dt.$$

   Use this and one of your differential equations to prove Kepler's second law: *The radius vector sweeps out equal area in equal time.*

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7. Ok. A quick reality check here. The two equations you should have for Problem 5 are $r'' - r(\theta')^2 = -GM/r^2$ and $2r'\theta' + r\theta'' = 0$. You did choose polar coordinates didn't you? Our goal in this problem is going to be to show that these equations imply that the planet must follow a path of the form
$$r = \frac{c}{1 + e\cos\theta} \qquad (*)$$
proving Kepler's first law: *The orbit of a planet is an ellipse.*
To be honest we're going to pull some TRICKS out of nowhere. We'll use them simply because they work. How they were thought of is not so important right now (wait until next semester). Here, I'll tell you what tricks to use.

   a) From the second equation (and what you did in Problem 6), show that $r^2\theta' = r_0 v_0$. Here r_0 and v_0 are the radius and speed, respectivly, at time $t = 0$

   b) Now we want to find a solution to the differential equation $r'' - r(\theta')^2 = -GM/r^2$. First get rid of all references to θ by substituting in the reult from part a).

   c) Now you should have an equation involving only constants, r and derivatives of r. Let's simplify this by making it an equation involving only a first derivative. We'll do that with a trick. Set $p = r'$. Show that
   $$r'' = p\frac{dp}{dr}.$$
   Hint: Think of p as $p(r(t))$, then use the chain rule.

   d) Rewrite the differential equation using only constants, r, p, and the derivative dp/dr.

   e) Move all the refernces to p to one side of the equation and all references to r to the other side. When this is possible we say the equation is *separable*. Now integrate both sides and conclude that
   $$(r')^2 = -\frac{r_0^2 v_0^2}{r^2} + \frac{2GM}{r} + C$$
   for some constant C. Find C by evaluating everything at time $t = 0$.

   f) Divide the left and right sides of this equation by the left and right sides of the equation $r^4(\theta')^2 = r_0^2 v_0^2$, respectively. Remember this guy from part a)?

   g) Now make the substitution $u = 1/r$. (So what is $du/d\theta$?) Also, to simplify the constants, set $h = \frac{GM}{r_0^2 v_0^2}$. After some algebra, show that
   $$\frac{du}{d\theta} = -\sqrt{(u_o - h)^2 - (u - h)^2}.$$
   (Why the negative root instead of the positive one?)

   h) Once again, we have a separable equation. Move everything involving u to one side and everything involving θ to the other. Then integrate both sides.
   Hint: use the trig substitution $u - h = (u_0 - h)\cos x$.

   i) Part h) gave you u as a function of θ. Now solve for $r = 1/u$. Finally, substitute back in the value of h. This should give you an equation of the form $(*)$. What are c and e?

8. Here, we seek to discover the period of the planet's orbit. This is the amount of time T required to complete one revolution of the orbit.

   a) Show that an ellipse of the form $r = c/(1 + e\cos\theta)$ has semimajor axis $a = c/(1-e^2)$, and semiminor axis $b = a\sqrt{1-e^2}$. Now write the area of an ellipse in terms of a and e.

   b) Recall that the rate of change of the area being swept out by the position vector is $A' = 1/2 r_0 v_0$. Integrate this (from time $t = 0$ to time $t =$ what?) to obtain another expression for the area of the ellipse.

   c) Set the two area equations you have equal to each other and solve for the period T. Square both sides. Replace $a(1-e^2)$ with the value of c which you discovered in Problem 7, and deduce Kepler's third law: *The square of the period is proportional to the cube of the mean distance to the sun. Specifically,*

$$T^2 = \frac{4\pi^2 a^3}{GM} .$$

9. Only Kepler's 3rd law actually assumed that the orbit was an ellipse. Actually, the equation $(*)$ from problem 7 only gives an ellipse when $0 \le e < 1$. This equation can also describe a parabola when $e = 1$ or a hyperbola when $e > 1$. For what values of v_0 (the initial velocity) would the orbit be

   a) an ellipse,

   b) a parabola, and

   c) a hyperbola?