WORKSHEET 31

1. In class Friday, you were given two unit vectors which are useful when using polar coordinates. At a point (r_0, θ_0), define

$$\mathbf{u}_r = (\cos\theta_0, \sin\theta_0) \qquad \mathbf{u}_\theta = (-sin\theta_0, \cos\theta_0).$$

a) Pick three points in the plane and draw both of these vectors. Why are these vectors named $\mathbf{u}_r$ and $\mathbf{u}_\theta$?

Another way to come up with basis vectors at a point (r_0, θ_0) is to define a vector $\frac{\partial}{\partial r}$ by parameterizing a path γ_r starting at (r_0, θ_0) at time $t = 0$, which keeps θ constant and has $r(t) = t$. Then set $\frac{\partial}{\partial r} = \gamma_r'(0)$. Define the vector $\frac{\partial}{\partial \theta}$ in the corresponding way as $\gamma_\theta'(0)$ for an appropraite path γ_θ.

b) Write down the paths γ_r and γ_θ in polar coordinates. Then convert these paths to Euclidean coordinates. Compute $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ in Euclidean coordinates.

c) Write $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$.

d) Find the dot products

$$\frac{\partial}{\partial r} \cdot \frac{\partial}{\partial r} \qquad \frac{\partial}{\partial r} \cdot \frac{\partial}{\partial \theta} \qquad \frac{\partial}{\partial \theta} \cdot \frac{\partial}{\partial \theta}.$$

2. Some believe that the positions of the planets at the time of birth influence the newborn. Others deride this and say that the gravitational force exerted on a baby by the obstetrician is greater than that exerted by the planets. To check this claim, calculate and compare the gravitational force exerted on a 6-kg baby

a) by a 70-kg obstetrician who is 1 m away,
b) by the massive planet Jupiter ($m = 2 \times 10^{27}$ kg) at its closest approach to earth (6×10^{11} m), and
c) by Jupiter at its greatest distance from earth (9×10^{11} m).

d) Is the claim correct?

3. Let $r(t)$ and $\theta(t)$ be polar coordinates for a particle moving in the plane. Derive formulas for the velocity and acceleration of the particle.
Hint: First find $\frac{d\mathbf{u}_r}{dt}$ and $\frac{d\mathbf{u}_\theta}{dt}$ using the chain rule. Then use the fact that the position is given by $r\mathbf{u}_r$, and differentiate.

4. Below, r and θ are the polar coordinates of a particle moving in the plane. For each motion, express the velocity and acceleration in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$.

$$r = a(1 - \cos\theta), \quad \frac{d\theta}{dt} = 3 \qquad r = a\sin 2\theta, \quad \frac{d\theta}{dt} = 2t \qquad r = a(1 + \sin t), \quad \theta = 1 - e^{-t}$$

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5. Without introducing coordinates, give a geometric argument for the validity of the equation

$$\frac{dA}{dt} = \frac{1}{2}\left|\mathbf{R} \times \frac{d\mathbf{R}}{dt}\right|,$$

where $\mathbf{R}$ is the position vector of a particle moving along a plane curve and dA/dt is the rate at which that vector sweeps out area.

6. A weather satellite is in geosynchronous orbit, hovering over Nairobi, which lies very close to the equator. If its orbit radius is increased by 1 km, at what rate and in what direction would its reference spot, which was formerly stationary, move across the earth's surface?

   Useful data: Earth's mass $= 5.975 \times 10^{24}$ kg, Equatorial radius of Earth $= 6378.533$ km, Gravitational constant $G = 6.6720 \times 10^{-11}\mathrm{Nm}^2\mathrm{kg}^{-2}$