WORKSHEET 28

1. Consider a path around the unit circle parametrized by $\boldsymbol{\sigma}$ on the interval $[0, 1]$ with

$$\boldsymbol{\sigma}(t) = \big(\sin(5\pi t - \pi), \cos(5\pi t - \pi)\big).$$

   a) Where does this path begin? Where does it end? Does it travel clockwise, counterclockwise, or sometimes one then the other?

   b) At time $t = 0$, what is the velocity vector? What is the speed? What is the acceleration vector?

   c) Is an object moving along this path accelerating at time $t = 0$? Is the speed increasing or decreasing?

2. Find the parametric equations

$$\boldsymbol{\gamma}(t) = \big(x(t), y(t)\big)$$

and a parameter interval for the motion of a particle which travels around the ellipse

$$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{16} = 1$$

once clockwise.

3. For the curves in Problems 1 and 2, find a formula for the **speed** of an object moving along the path.

4. **Curvature.** The *curvature* of a path $\boldsymbol{\sigma}(t)$ is defined to be the rate at which the unit tangent vector is changing with respect to arc length. In this problem, we develop this idea more explicitly.

   The unit tangent vector is defined to be

$$\mathbf{T} = \frac{\boldsymbol{\sigma}'(t)}{|\boldsymbol{\sigma}'(t)|}$$

   where $|\boldsymbol{\sigma}'(t)|$ denotes the length of $\boldsymbol{\sigma}'(t)$ or the speed at t.

   a) Why is $\mathbf{T}$ called the unit tangent vector?

   Now we define the *curvature vector* to be

$$\boldsymbol{\kappa} = \frac{1}{|\boldsymbol{\sigma}'(t)|}\dot{\mathbf{T}}.$$

   The curvature vector points in the direction in which $\mathbf{T}$ is turning, perpendicular to $\mathbf{T}$.

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b) Does the curvature vector always point in the same direction as the acceleration vector?

We define the *scalar curvature* κ to be the length of the curvature vector. That is

$$\kappa = |\boldsymbol{\kappa}|.$$

For a circe of radius r, the curvature vector $\boldsymbol{\kappa}$ points toward the center of the circle and has length $\kappa = 1/r$. For a general curve, the best approximating (or *osculating*) circle has radius $1/\kappa$.

c) Without doing any computations, answer the following questions:
i) What is the scalar curvature κ at any point on the path in problem 1?
ii) Is κ constant for the curve in problem 2?
iii) If not, where is it the greatest? the least?

d) Prove that the curvature vector is always perpendicular to the tangent vector.

e) When will the curvature vector and the acceleration vector point in the same direction?