WORKSHEET 23

1. a) Let $\mathbf{v}$ be a vector in $\mathbb{R}^2$. Determine the set of all vectors $\mathbf{w}$ such that $\mathbf{v} \cdot \mathbf{w} = 0$.

   b) Let $\mathbf{v}$ be a vector in $\mathbb{R}^3$. Determine the set of all vectors $\mathbf{w}$ such that $\mathbf{v} \cdot \mathbf{w} = 0$.

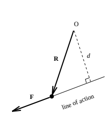
   c) Let $\mathbf{v}$ be a vector in $\mathbb{R}^3$. Determine the set of all vectors $\mathbf{w}$ such that $\mathbf{v} \times \mathbf{w} = 0$.

2. Describe what happens to the vectors

$$(\cos t, \sin t) \qquad \text{and} \qquad (\cos t, -\sin t)$$

as the real parameter t varies. For what values of t is the area of the parallelogram spanned by these vectors maximized?

3. In mechanics, the *moment M of a force* $\mathbf{F}$ *about a point O* is defined to be the magnitude of $\mathbf{F}$ times the perpendicular distance d from O to the line of action of $\mathbf{F}$. The *vector moment* $\mathbf{M}$ is the vector of magnitude M whose direction is perpendicular to the plane of O and $\mathbf{F}$ determined by the right-hand rule. Show that $\mathbf{M} = \mathbf{R} \times \mathbf{F}$, where $\mathbf{R}$ is any vector from O to the line of action of $\mathbf{F}$.
(Hint: What information is contained in the vector $\mathbf{R} \times \mathbf{F}$?)



4. The *angular velocity* $\boldsymbol{\omega}$ of rotation of a rigid body has direction equal to the axis of rotation and magnitude equal to the rate of spinning in radians per second. The sense of $\boldsymbol{\omega}$ is determined by the right-hand rule.

   a) Let $\mathbf{r}$ be a vector from the axis to a point P on the rigid body. Show that the quantity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ is the velocity of P, as shown below.

   b) Interpret the result for the rotation of a carousel about its axis, with P a point on the circumference.

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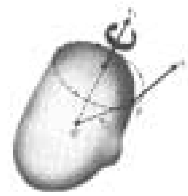

5. **The Precessing Top.** Consider a couple of young children, Billy and Janie, in the midst of their carefree childhood days.... Their inquisitive nature and hunger for discovery has yet to be crushed by years of dull rote work forced on them by "educators" bitterly vindicating their own extinguished excitement over knowledge. They are playing with a top. Billy sets the top in motion at a slight angle from vertical, and Janie asks why the top doesn't fall over. She picks up the top and sets it down at the same angle only not spinning, and the top falls. "See," she says, "when it's not spinning it just falls over!" Billy snidely responds, "That's because of gravity. Don't you know anything about physics?" Janie, in her infinite patience tries to explain once more, "But that doesn't explain why the top doesn't fall when it is spinning." She sets the top in motion again at a slight angle, and they both are shocked when they notice something even stranger. "It's precessing!" shouted Billy. He's right! The axis of the top sweeps out a slow circle. Eager to figure out why this was occurring, they go to their Encyclopaedia Britannica and discover Newton's Second Law, $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ is a force vector, m is the mass of an object, and $\mathbf{a}$ is the acceleration vector of that object due to the application of the force $\mathbf{F}$. Janie pulls out some scratch paper, and says, "Look! If we define *torque* to be the cross product of the position vector $\mathbf{r}$ with the force vector $\mathbf{F}$, and define *angular momentum* to be mass times the cross product of the position vector $\mathbf{r}$ with the velocity vector $\mathbf{v}$..." At the same time she writes down

$$\text{torque: } \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

$$\text{angular momentum: } \mathbf{l} = m(\mathbf{r} \times \mathbf{v})$$

"Wait!" interrupts Billy, "Why are you defining these weird things?" to which Janie responds "Because look what happens when you differentiate the equation defining angular momentum, then substitute Newton's Second Law. You get

$$\boldsymbol{\tau} = \frac{d\mathbf{l}}{dt}\,.''$$

a) Prove this relation by proceeding as Janie suggests.

"Aha!" exclaims Billy, "This explains the precession of the top!"

b) Using the point where the top touches the floor as the origin, determine the direction in which the angular momentum vector of the top points?
Hint: Each point in the top contributes. Pick a point and draw the velocity and position vectors for that point.

c) In which direction does the force vector due to gravity point?

d) In which direction does the torque vector point?

e) Use Janie's equation to determine what effect the force of gravity must have on the axis of rotation of the top.