

WORKSHEET 14

1.
 - a) Graph the function $f(x) = \sin x$ on a set of axes that is at least half the size of a page of notebook paper.
 - b) Find the equation of the tangent line to f at $x = 0$. Graph it on the same set of axes. Notice that near $x = 0$, this gives a reasonable approximation for f . Most likely you have seen the approximation $\sin x \approx x$ for x small. (In physics class?)
 - c) Perhaps we could approximate $f(x) = \sin x$ more accurately with a parabola than with a line. Find the equation of the parabola which goes through $(0, 0)$, has the same slope as $\sin x$ at $x = 0$, **and** which has the same second derivative as $\sin x$ at $x = 0$.
 - d) Now find the equation of a cubic which goes through $(0, 0)$ and which also has the same value of first, second, **and** third derivative as $f(x) = \sin x$ at $x = 0$.
2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function which is infinitely differentiable. (What does that mean?) Consider the n^{th} degree polynomial

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n.$$

Suppose that the values of each of the first n derivatives of f at $x = 0$

$$f'(0), f''(0), f'''(0), \dots, f^{(n)}(0)$$

are the same as the values of the respective derivatives of the above polynomial. Find the values of the coefficients a_i in terms of the derivatives of f . This is called the n^{th} order Taylor polynomial of f at 0.

(Hint: Evaluate the polynomial and its first n derivatives at $x = 0$.)

3. **Definition.** Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function. The Taylor series of f at $x = 0$ is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

- a) Compute the Taylor series at $x = 0$ for the following functions:

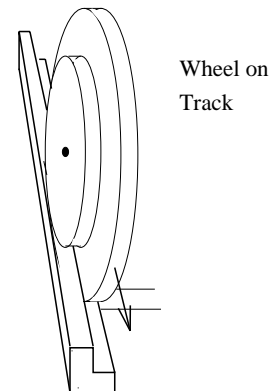
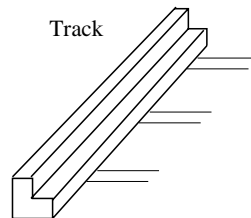
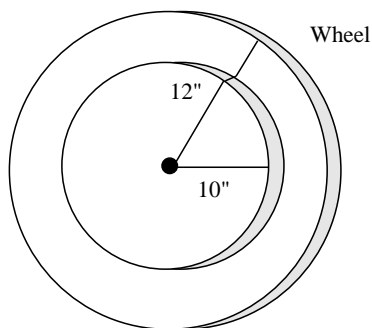
$$\text{i) } f(x) = \sin x \quad \text{ii) } f(x) = \cos x \quad \text{iii) } f(x) = e^x \quad \text{iv) } f(x) = \frac{1}{1-x}$$

- b) For which values of x do these series converge?

4.
 - a) Which polynomials are odd functions?
 - b) If f is an odd function, are its Taylor polynomials necessarily odd? Explain.
 - c) Which polynomials are even functions?
 - d) If f is an even function, are its Taylor polynomials necessarily even? Explain.
5. For values x where the series converges, define

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{x+3}{5} \right)^n .$$

- a) Find a closed form expression for $f(x)$. (i.e., one which does not involve an infinite sum.)
 - b) For what values of x does the series defining $f(x)$ converge?
 - c) On the interval where the series defining $f(x)$ converges, graph $f(x)$. Looking at the graph, why do you think the interval of convergence ends where it does?
6. A train wheel has two rims as shown below and is designed to run along a bi-level track. The outer (smaller) rim has radius 10 inches while the inner (larger) rim has radius 12 inches.



- a) How far does the outer rim roll over the course of one rotation of the wheel?
- b) How far does the inner rim roll over the course of one rotation of the wheel?
- c) How far does the train travel over the course of one rotation of the wheel?