

WORKSHEET 13

1. A *power series* in x is a series of the form

$$S(x) = \sum_{n=0}^{\infty} a_n x^n.$$

where x is a real number and $\{a_n\}$ is a sequence of real numbers. We may then ask the question:

For a given value of x , does the series $S(x)$ converge?

Suppose that for some value c , the series $S(c)$ does converge.

- a) Argue that there is a number N such that for all $n > N$, we have

$$|a_n c^n| < 1.$$

(Hint: Use the divergence test.)

- b) Show that if $|x| < |c|$ and $n > N$ then

$$|a_n x^n| < \left| \frac{x^n}{c^n} \right|.$$

- c) Prove that if $|x| < |c|$, then the series $S(x)$ converges **absolutely**.
- d) Consider the three sets of values of x where $S(x)$ converges absolutely, where it converges conditionally, and where it diverges. Sketch on a number line what these sets can look like. Explain what this means as far as testing the convergence of power series.
2. For values x where the series converges, define

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{x+3}{5} \right)^n.$$

- a) Find a closed form expression for $f(x)$. (i.e., one which does not involve an infinite sum.)
- b) For what values of x does the series defining $f(x)$ converge?
- c) On the interval where the series defining $f(x)$ converges, graph $f(x)$. Looking at the graph, why do you think the interval of convergence ends where it does?

3. The polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ could also be written

$$f(x) = \sum_{k=0}^n a_k x^k.$$

What connection, if any, is there between the latter and the series $\sum_{k=0}^{\infty} a_k x^k$?

- Write the series representation for the first four derivatives of $f(x)$.
(e.g. $f'(x) = \sum_{k=1}^n k a_k x^{k-1}$)
 - Solve for the coefficients $a_0, a_1, a_2, a_3,$ and a_4 .
(Hint: Set $x = 0$ and solve for a_i in terms of the i^{th} derivative $f^{(i)}(0)$.)
 - Can you generalize your findings to give an expression for the k^{th} coefficient a_k .
 - Write $f(x)$ as a series using the expressions for the coefficients obtained in parts b) and c). This is called the *Maclaurin series* for $f(x)$.
 - Apply steps a)-e) above to $f(x) = \sum_{k=0}^n a_k (x-a)^k$, where a is a real number, to obtain the *Taylor series* representation of $f(x)$.
 - We say the Taylor series is a series representation of $f(x)$ expanded about the real number a . So, the Maclaurin series is simply a Taylor series expanded about which real number?
4. Compute the first four derivatives of the following functions. Then write a formula that gives their n^{th} derivative $f^{(n)}(x)$ and their values $f^{(n)}(0)$ at $x = 0$:

a) $f(x) = \sin x$

b) $f(x) = \cos x$

c) $f(x) = e^x$

5. **The Monty Hall Problem.** Congratulations! You've been selected as a contestant on "Let's Make a Deal"! Behind one of the three doors in front of you is a million dollar cash prize, behind another is a five year supply of Spam, and behind the third is a pet goat. Whichever door you choose, Monty Hall will open one of the other two doors to reveal either the goat or the Spam. He will then give you the opportunity to switch your choice to the other closed door if you wish. Are you more likely, less likely, or equally likely to win the million dollars if you switch?

Bonus question: Exactly how much Spam constitutes a "five year supply"?