WORKSHEET 11

1. Define a function $f(x)$ by the following rule: for any $x \in \mathbb{R}$, set

$$f(x) = \sum_{n=0}^{\infty} x^n.$$

   a) I was intentionally careless in defining this function. What is its domain?

   b) Find another, simpler way of expressing this function. Find its derivative, and then graph f.

   c) What is $f(-1)$? Use both your closed form expression for f and the series definition originally given.

   d) Where are the vertical asymptotes of f? For what values of x does the series above converge?

2. For each value of $x \in \mathbb{R}$, consider the series

$$\sum_{n=1}^{\infty} x(1-x)^n \quad = \quad x(1-x) + x(1-x)^2 + x(1-x)^3 + \cdots$$

   a) For what values of x does this series converge? For these values, denote the sum of the series by $F(x)$.

   b) What is the domain of F?

   c) What is the value of

$$\lim_{x \to 0} F(x)?$$

   d) Factor out an $x(1-x)$ from every term in the series. Note that the other factor is then a geometric series. Use the formula for a geometric series to write a closed form expression for $F(x)$.

   e) Use the expression obtained in part c) to compute

$$\lim_{x \to 0} F(x)$$

   and compare with your answer from part b).

   f) Which is the correct limit?

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3. Consider the series
$$g(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}.$$

   a) For what values of x does $g(x)$ converge? Where it does converge, is it absolutely or conditionally?

   b) Find $g'(x)$.

   c) For what values of x does your answer to part c) make sense? That is, where does it converge?

4. Consider the series
$$\sum_{n=1}^{\infty} [(n+1)x - n]x^n.$$
   Call this sum $F(x)$ for values of x for which it converges.

   a) What is
$$\lim_{x \to 1} F(x)?$$

   b) Simplify the expression for this infinite series algebraically and write out the first five terms. Does anything cancel? Now what do you think the limit from part a) is?

5. Now let us consider the limit
$$\lim_{n \to \infty} \int_0^1 n^2 x e^{-nx}\, dx.$$

   a) For a fixed value of x, what is
$$\lim_{n \to \infty} n^2 x e^{-nx}?$$
   What does this suggest is the limit of the integral?

   b) Let $u = nx$ and $du = n\,dx$. Rewrite the integral above with this substitution. Don't forget to change the limits of integration. Solve the integral, and take the limit as $n \to \infty$.