

WORKSHEET 10

1. Carefully state the following tests for convergence or divergence of a series. Give examples of series to which the tests could be applied in determining convergence or divergence.

- a) *Divergence Test* b) *Geometric Series Test* c) *Comparison Test*
 d) *Integral Test* e) *Alternating Series Test* f) *p - Series Test*

2. a) State the *comparison test* for the convergence or divergence of a series. Apply this test to the following series:

$$\text{i) } \sum_{n=1}^{\infty} \frac{4n}{n^3 + n} \quad \text{ii) } \sum_{n=1}^{\infty} \frac{4n}{n^3 - n} \quad \text{iii) } \sum_{n=1}^{\infty} \frac{4n^2}{n^3 + n}$$

- b) In which of these examples was the application of the comparison test more difficult. Why?
- c) For each of these series, give a quick intuitive argument which may not be completely rigorous but which sidesteps the difficulties you encountered. (Hint: What is the **general** behavior of the terms as $n \rightarrow \infty$?) Why is this argument not rigorous?
- d) Determine $\lim_{n \rightarrow \infty} n^2 a_n$ where a_n is the n^{th} term of series ii) in part a) above.
- e) Use your answer to c) to show that there must be an integer N such that

$$a_n < \frac{1}{n^2}$$

for all $n \geq N$. What does this tell you about the convergence of the series $\sum a_n$?

- f) Now we will formalize this argument into a process that will allow us to apply the comparison test more easily in many cases. Suppose that $\sum_{n=1}^{\infty} p_n$ is a series of positive terms to be tested for convergence or divergence. Suppose, also, that we have another series $\sum_{n=1}^{\infty} c_n$ which we know converges and that the limit

$$\lim_{n \rightarrow \infty} \frac{p_n}{c_n} = a$$

exists. Show that there must be an integer N such that

$$p_n < (a + 1)c_n$$

for all $n \geq N$. What does this tell you about the convergence of the series $\sum p_n$?

In part f) you proved part of the *limit-comparison test*. Fully stated, it says:

Let $\sum_{n=1}^{\infty} p_n$ be a series of positive terms to be tested for convergence or divergence.

- i) Let $\sum_{n=1}^{\infty} c_n$ be a convergent series of positive terms. If $\lim_{n \rightarrow \infty} p_n/c_n$ exists, then $\sum_{n=1}^{\infty} p_n$ also converges.
- ii) Let $\sum_{n=1}^{\infty} d_n$ be a divergent series of positive terms. If $\lim_{n \rightarrow \infty} p_n/d_n$ exists and is not zero or if it is infinite, then $\sum_{n=1}^{\infty} p_n$ also diverges.
- g) Apply the limit-comparison test to each of the series in part a).

3. The first thing you should do (at least mentally if not on paper) when trying to determine whether a series $\sum_{n=1}^{\infty} a_n$ converges or diverges is to consider the limit

$$\lim_{n \rightarrow \infty} a_n.$$

The primary information you gain here is that if the limit is not zero, then the series diverges, and if the limit is zero, then it is **possible** that the series converges. (Why do you think I stressed the word “possible”?)

In the case that the limit is zero (so that we must explore further tests), there is also some secondary information that can be gained from considering this limit. Specifically, noticing the general behavior of the terms as $n \rightarrow \infty$ can often suggest what series to use for a comparison or limit-comparison test. For example, the terms of the series in Problem 2 a) tend to behave as $4/n^2$, $4/n^2$, and $4/n$ respectively, as n tends to infinity.

- a) For each of the following series, find the limit of the sequence of terms.

$$\begin{array}{lll} \text{i) } \sum_{n=1}^{\infty} \frac{4n^2 - 7n + \pi}{\frac{1}{97}n^3 - 80n^2 - 10^5n - 1} & \text{ii) } \sum_{n=1}^{\infty} \frac{\sin 1/n}{\ln n} & \text{iii) } \sum_{n=1}^{\infty} 6n^{38} e^{.005n} \\ \text{iv) } \sum_{n=1}^{\infty} n^{-\frac{n+1}{n}} & \text{v) } \sum_{n=1}^{\infty} n^{-\frac{n}{n+1}} & \text{vi) } \sum_{n=1}^{\infty} \frac{\ln n}{n^{1.001}} \end{array}$$

- b) For each of the series given in part a), describe the limiting behavior of the terms.
- c) Use your answer to part b) to apply either the comparison test or limit-comparison test to each of these series.

4. Suppose $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n^2$ converges. Show that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

(Hint: compare $\frac{a_n}{n}$, a_n^2 , $\frac{1}{n^2}$)