WORKSHEET 9

1. Test for convergence or divergence.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{2n^3-1}$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \ln n}$

e) $\displaystyle\sum_{k=1}^{\infty} e^{-k^2}$ f) $\displaystyle\sum_{n=1}^{\infty} \frac{4n^2-n+3}{n^3+2n}$ g) $\displaystyle\sum_{k=1}^{\infty} ke^{-k^2}$ h) $\displaystyle\sum_{n=1}^{\infty} \frac{n+\sqrt{n}}{2n^3-1}$

i) $\displaystyle\sum_{i=1}^{\infty} \sin^3\left(\frac{1}{i}\right)$ j) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2+3}$ k) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{3n^2+2n+5}$ l) $\displaystyle\sum_{k=1}^{\infty} \frac{\log k}{k}$

2. Give an example of each of the following. If this is not possible, state why.

a) A function f such that $\displaystyle\int_1^{\infty} f(x)\,dx$ diverges, but $\displaystyle\lim_{x\to\infty} f(x) = 0$.

b) A function f such that $\displaystyle\int_1^{\infty} f(x)\,dx$ converges, but $\displaystyle\lim_{x\to\infty} f(x) \neq 0$.

c) A series $\displaystyle\sum_{n=1}^{\infty} a_n$ which diverges, but $\displaystyle\lim_{n\to\infty} a_n = 0$.

d) A series $\displaystyle\sum_{n=1}^{\infty} a_n$ which converges, but $\displaystyle\lim_{n\to\infty} a_n \neq 0$.

3. a) On the same set of axes, sketch the graphs of the functions $1/x^2$ and $1/(x^2-1)$ over the interval $[2, \infty)$.

Looking at the graph, do you think that $\int_2^{\infty} 1/(x^2-1)\,dx$ converges or diverges? Why?

Does the **Comparison Test** apply here?

b) The *Ratio Test* for improper integrals states that if $f(x)$ and $g(x)$ are two positive functions such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty$$

then $\int_a^{\infty} f(x)\,dx$ and $\int_a^{\infty} g(x)\,dx$ either both converge or both diverge.

Apply the Ratio Test to determine whether $\int_2^{\infty} \frac{dx}{x^2-1}$ converges or diverges.

c) Prove the Ratio Test.
(Hint: There is a point $x_0 > a$ such that whenever $x > x_0$,

$$\frac{L}{2} < \frac{f(x)}{g(x)} < 2L.$$

Why? Now multiply this entire equation by $g(x)$ and use the Comparison Test.)

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4. Here is an incorrect statement of the integral test:

   Theorem. Let $f\colon [1,\infty)$ be a function and let $a_n = f(n)$. Then
   (i) if $\int_1^\infty f(x)\,dx$ is convergent, so is the series $\sum_{n=1}^\infty a_n$, and
   (ii) if $\int_1^\infty f(x)\,dx$ is divergent, so is the series $\sum_{n=1}^\infty a_n$.

   What is missing here are some conditions on the function f. In other words, the integral tes cannot be applied to just any function. What are the missing conditions? Come up with some functions f for which the above "Theorem" is wrong.

5. Find the sum of the series:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

   Note that the general term a_n is:

$$a_n = \frac{1}{n(n+1)}\,.$$

   What is the partial fractions decompostion of a_n? What does the series look like if we write the terms that way?