WORKSHEET 7

1. **Definition.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that the sequence *converges* if there is some real number L, called the *limit* of the sequence, such that for any $\varepsilon > 0$, there is a number N such that

$$|a_n - L| < \varepsilon$$

whenever $n \geq N$. We sometimes express this by writing

$$\lim_{n\to\infty} a_n = L.$$

If there is no such number L, then we say that the sequence $\{a_n\}$ *diverges*.

   a) Draw a picture of the real number line which explains this definition of a convergent sequence. Take turns explaining it to each other to make sure that everyone in your group has a clear understanding.

   b) Prove that the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges.

   c) Give an example of an infinite sequence which does not converge. Justify your answer.

2. **The Squeeze Theorem.** Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are sequences which converge to the same limit L, and that $\{b_n\}_{n=1}^{\infty}$ is a third sequence such that

$$a_n \leq b_n \leq c_n \qquad \text{for all } n.$$

Prove that

$$\lim_{n\to\infty} b_n = L.$$

3. Prove that the sequence

$$0.9, \quad 0.99, \quad 0.999, \quad 0.9999, \quad 0.99999, \quad 0.999999, \quad \ldots$$

converges. (to what limit?)

4. Rewrite the following series using a Riemann sum:

   a) $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots =$

   b) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \cdots =$

   c) $\frac{101}{3} + \frac{102}{10} + \frac{103}{29} + \frac{104}{66} + \frac{105}{127} + \cdots =$

   d) $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \cdots =$

   e) $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243} + \cdots =$

Mike Oehrtman

5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and define a new sequence $\{S_n\}_{n=1}^{\infty}$ in the following way:

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ S_4 &= a_1 + a_2 + a_3 + a_4 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n. \end{aligned}$$

This is called *the sequence of partial sums of* $\{a_n\}$.

a) Express the sequence in Problem 3 as a sequence of partial sums. What are $\{a_n\}$ and $\{S_n\}$ in this case? What is

$$\lim_{n\to\infty} a_n$$

for this sequence of a_n?

b) Now let $\{a_n\}$ be any sequence and suppose that the corresponding sequence of partial sums $\{S_n\}$ happens to converge. Write down what that means in terms the definition in Problem 1.

c) Prove that the sequence $\{a_n\}$ converges to zero.

Hint: First write down what it is you want to show in terms of the definition of convergence. This will keep you on track, and you can refer back to it if you start to get confused. Now notice that (as silly as this may seem) you can write $|a_n|$ as

$$|a_n + S_n - S + S - S_n|.$$

Think of a more helpful way to write $a_n + S_n$ in this expression, then use the fact (called the *triangle inequality*) that

$$|x + y| \leq |x| + |y| \qquad \forall x, y \in \mathbb{R}.$$

6. You are driving along Highway 50 just outside of Canon City, Colorado eager to get back to Austin for your ESP class the next day. Just after you pass the Royal Gorge (Where does that bridge go to anyway?) and are starting up a steep hill, your fuel pump begins to fail. You begin to ascend the hill, but due to the low pressure in the pump, begin rolling downhill. You manage to stop descending only after rolling downhill half the initial distance ascended. You start up again and ascend one third of the initial distance upward before your car acts up again and forces you downhill one fourth of the initial distance. This continues so that at the n^{th} stage you are either rolling downhill one n^{th} of the initial distance or moving uphill one n^{th} of the initial distance. If this were to continue indefinitely, what would happen to your position (that is, the **net** distance traveled)? Prove or disprove that the *total* distance traveled is finite. Do you ever make it up the hill and back to class?

7. A bear walks one mile due south, then one mile due east, and then one mile due north. He ends up at exactly the same spot where he started. What color is the bear?