WORKSHEET 6

1. **Zeno's Tortoise.** You all know the story of the tortoise and the hare. The actual race many years ago between these two well known rivals was two kilometers long (animals are smart enough to know the metric system is much better than the English system) and the tortoise was given a half kilometer head start. The hare could run two kilometers per hour compared to the tortoise's one kilometer per hour. (This is actually a blazing speed for a tortoise, and you have to remember that he had won many preliminary races before finding himself pitted against the hare in this final round.) On the day of the race there happened to be a slug in the audience who was saying to anyone who would listen that the race was futile, and the hare had no chance of winning. His reasoning went like this:

> By the time the hare reaches the 1/2 km mark (where the tortoise starts), the tortoise will already have progressed some distance. Then, in the time it takes the hare to make up that distance, the tortoise will have progressed to a further location, maintaining his lead. Again, while the hare is running to that location, the tortoise will have moved still farther, leaving yet another gap. In this way, the tortoise will always remain in front.

   a) What is wrong with the slug's reasoning?

   b) At what point does the hare actually overtake the tortoise? (Sorry if you had your money on the tortoise, but he does indeed lose.)

   c) The first leg of the slug's description of the race is 1/2 km. How far is the second? (i.e. the distance the tortoise covers while the hare is running the first leg?) How far is the third leg? the fourth?... the n^{th}?

   How many of these so called legs will there be according to the slug's description of the race?

   d) What is the value of the infinite sum

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}+\cdots?$$

2. a) Express the following decimals as fractions:

   (i) $.1111111\ldots = 0.\overline{1}$ (iii) $2.017171717\ldots = 2.0\overline{17}$

   (ii) $.060606060606\ldots = 0.\overline{06}$ (iv) $.9999999\ldots = 0.\overline{9}$

   b) An *infinite geometric series* is a sum of the form:

$$S = a + ar + ar^2 + \cdots + ar^n + \cdots$$

   Express the decimals in part a) as a geometric series (i.e. what are a and r?).

   c) Find a formula for S, when $|r| < 1$.

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3. Does the infinite sum
$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$$
have an answer?

4. Suppose that $f(x)$ and $g(x)$ are continuous and $0 \le f(x) \le g(x)$ for all $x \in [a,\infty)$. For each of the following statements on the left, match with a non-vacuous conclusion on the right. (An example of a vacuous match would be to pair a) with iii).)

a) If $\int_a^\infty g(x)\,dx$ converges then...
b) If $\int_a^\infty g(x)\,dx$ diverges then...
c) If $\int_a^\infty f(x)\,dx$ converges then...
d) If $\int_a^\infty f(x)\,dx$ diverges then...

i) $\int_a^\infty f(x)\,dx$ converges.
ii) $\int_a^\infty f(x)\,dx$ diverges.
iii) $\int_a^\infty g(x)\,dx$ converges.
iv) $\int_a^\infty g(x)\,dx$ diverges.
v) not conclusive.

Now, give explanations/reasons or counterexamples which illustrate the mathces you made above.

5. a) Show that $\lim\limits_{c\to\infty} \int_{-c}^{c} x\,dx$ exists. **Find it.**

   b) Explain why the limit in a) exists however $\int_{-\infty}^{\infty} x\,dx$ diverges.

6. Let $f\colon [0,\infty) \to \mathbb{R}$ be a function with $f(0) = 2$ and $\lim_{x\to\infty} f(x) = 3$. Find the following:

   a) $\lim\limits_{b\to\infty} \frac{1}{b}\int_0^b f(x)\,dx$ b) $\int_0^\infty f(x)\,dx$ c) $\lim\limits_{b\to 0} \frac{1}{b}\int_0^b f(x)\,dx$

7. **The Hilbert Hotel.** You own a magical hotel in which there are infinitely many rooms numbered

$$1,\quad 2,\quad 3,\quad 4,\quad 5,\quad 6,\quad 7,\quad 8,\quad 9,\quad 10,\quad 11,\quad 12,\quad \ldots$$

One night when the hotel is full, someone arrives wanting a room. As the owner and manager you are allowed to move guests around, as long as you tell each person the exact number of their new room. Can you accomodate the new arrival? (Call the guest in room 1, G_1, the guest in room 2, G_2, etc.)

Later, 25 people show up. Can you accomodate them?

What if infinitely many people show up who are named (for your convenience) N_1 (for new guest 1), N_2, $N_3, \ldots$ Notice each new guest is labeled with a natural number.

What if each of the new guests were labeled with a **real** number?