Math 300
Supplementary Examples

Induction

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Example 1. Every positive integer n has a factorization $n = 2^\ell m$, for some $\ell \in \mathbb{N}$ and some odd positive integer m.
proof: (induction on n). If $n = 1$ then we may take $\ell = 0$ and $m = 1$. Now suppose there is a positive integer k such that the result is true for all positive integers $n < k$ and consider $n = k$. If k is odd, then we may take $\ell = 0$ and $m = k$. If k is even, then there is an integer r such that $k = 2r$. Since k is positive, r is positive. Also, $r < k$, so we may apply the induction hypothesis to conclude that there is a natural number t and an odd positive integer s such that $r = 2^t s$. But this implies that $k = 2 \cdot 2^t s = 2^{t+1} s$. □

Example 2. Let F_n denote the n^{th} Fibonacci number. Then F_n is even if and only if $3 \mid n$.
proof: ($\leftarrow$). Suppose $3 \mid n$, say $n = 3j$, where $j \in \mathbb{N}$. We induct on j. If $j = 0$, then $n = 0$ and $F_n = 0$, which is even. Now assume there is some positive integer k such that the result is true for all natural numbers $j < k$ and consider $j = k$. We have

$$F_{3k} = F_{3k-1} + F_{3k-2} = F_{3k-2} + F_{3k-3} + F_{3k-2} = 2F_{3k-2} + F_{3(k-1)}.$$

The induction hypothesis tells us that $F_{3(k-1)}$ is even, and clearly $2F_{3k-2}$ is even, so F_{3k} is even.

($\rightarrow$). Suppose $3 \nmid n$. Then $n = 3j + r$, where $j \in \mathbb{N}$ and $r \in \{1, 2\}$. We must show that F_n is odd. We do this by using induction on j to show that F_{3j+1} and F_{3j+2} are always odd for any j. If $j = 0$ then n is either 1 or 2, so $F_n = 1$, which is odd. Now suppose there is a positive integer k such that the result is true for all natural numbers $j < k$ and consider $j = k$. Then $F_{3k+1} = F_{3k} + F_{3k-1} = F_{3k} + F_{3(k-1)+2}$. We showed earlier that F_{3k} is even. By the induction hypothesis, $F_{3(k-1)+2}$ is odd. Hence F_{3k+1} is the sum of an even integer and an odd integer, so is odd. Similarly, $F_{3k+2} = F_{3k+1} + F_{3k}$ is the sum of an odd integer and an even integer, so is odd. □

Example 3. Let F_n denote the n^{th} Fibonacci number. If m and n are natural numbers and $m \mid n$, then $F_m \mid F_n$. *(Note that example 2 and example 3 are related — how?)*
proof: Let $m \in \mathbb{N}$. If $n \in \mathbb{N}$ is any multiple of m, then $n = mj$ for some natural number j. We use induction on j to show that $F_m \mid F_{mj}$ for any such j. This is clear if $j = 0$. Suppose there is some positive integer k such that the result is true for all natural numbers $j < k$ and consider $j = k$. Then

$$\begin{aligned}
F_{mk} &= F_{m(k-1)+m} \\
&= F_{m(k-1)-1}F_m + F_{m(k-1)}F_{m+1} && \text{by problem 6.4.7a} \\
&= F_{m(k-1)-1}F_m + F_m G\, F_{m+1} && \text{for some integer } G \text{, by the induction hypothesis} \\
&= F_m\big(F_{m(k-1)-1} + F_{m+1}G\big)
\end{aligned}$$

so $F_m \mid F_{mk}$ as needed. □

Example 4. Let F_n denote the n^{th} Fibonacci number. Show, for all $m \geq 1$, that $F_{2m-1} = \sum_{j=0}^{m-1} \binom{2m-j-2}{j}$ and $F_{2m} = \sum_{j=0}^{m-1} \binom{2m-j-1}{j}$.

proof: (Induction on m.) For $m = 1$, we compute: $\sum_{j=0}^{m-1} \binom{2m-j-2}{j} = \binom{0}{0} = 1 = F_1$ and $\sum_{j=0}^{m-1} \binom{2m-j-1}{j} = \binom{1}{0} = 1 = F_2$, so the equations hold for $m = 1$. Now suppose there is some positive integer k such that the equations are true for all positive integers $m < k$ and consider $m = k$. We have

$$\begin{aligned}
F_{2k-1} &= F_{2k-2} + F_{2k-3} \\
&= \sum_{j=0}^{k-2} \binom{2(k-1)-j-1}{j} + \sum_{j=0}^{k-2} \binom{2(k-1)-j-2}{j} \qquad \text{by the induction hypothesis} \\
&= \sum_{j=0}^{k-2} \binom{2k-j-3}{j} + \sum_{j=0}^{k-2} \binom{2k-j-4}{j} \\
&= \sum_{j=0}^{k-2} \binom{2k-j-3}{j} + \sum_{j=1}^{k-1} \binom{2k-j-3}{j-1} \\
&= \binom{2k-3}{0} + \binom{k-2}{k-2} + \sum_{j=1}^{k-2} \left[\binom{2k-j-3}{j-1} + \binom{2k-j-3}{j} \right] \\
&= 1 + 1 + \sum_{j=1}^{k-2} \binom{2k-j-2}{j} \\
&= \sum_{j=0}^{k-1} \binom{2k-j-2}{j}
\end{aligned}$$

and

$$\begin{aligned}
F_{2k} &= F_{2k-1} + F_{2k-2} \\
&= \sum_{j=0}^{k-1} \binom{2k-j-2}{j} + \sum_{j=0}^{k-2} \binom{2(k-1)-j-1}{j} \qquad \text{by the above computation} \\
&= \sum_{j=0}^{k-1} \binom{2k-j-2}{j} + \sum_{j=1}^{k-1} \binom{2k-j-2}{j-1} \\
&= \binom{2k-2}{0} + \sum_{j=1}^{k-1} \left[\binom{2k-j-2}{j-1} + \binom{2k-j-2}{j} \right] \\
&= 1 + \sum_{j=1}^{k-1} \binom{2k-j-1}{j} \\
&= \sum_{j=0}^{k-1} \binom{2k-j-1}{j}
\end{aligned}$$

so both equations hold for $m = k$. $\square$

Example 5. Let a and b be positive integers. There exist integers x, y such that $\gcd(a, b) = ax+by$.
proof: Let $\mathcal{C}$ denote the set of all positive integers that can be written as $ax + by$ for some $x, y \in \mathbb{Z}$. Since $a \in \mathcal{C}$, we know $\mathcal{C}$ is non-empty. By the well-ordering property of $\mathbb{Z}_+$, we know that $\mathcal{C}$ contains a smallest element, d say. Since $d \in \mathcal{C}$, there are integers x_0, y_0 such that $d = ax_0 + by_0$. Now $a \in \mathcal{C}$, so $d \leq a$. Similarly $b \in \mathcal{C}$, so $d \leq b$. Divide a by d, to obtain $a = dq+r$ for integers q and r such that $0 \leq r < d$. Now $a-r = qd = q(ax_0+by_0) = qax_0+qby_0$, so $r = a-qax_0-qby_0 = a(1-qx_0)+b(-qy_0)$. Thus, if r is positive then $r \in \mathcal{C}$, contradicting the minimality of d in $\mathcal{C}$. It follows that $r = 0$, whence $d \mid a$. A similar argument shows $d \mid b$. Hence d is a common divisor of a and b. To show that d is the greatest common divisor of a and b, we must show that any other common divisor of a and b is smaller than d. Suppose n is any common divisor of a and b. We must show $n \leq d$. This will follow if we show $n \mid d$. But this is clear since $n \mid a$ and $n \mid b$, so that we must have $n \mid (ax_0 + by_0)$. □

Exercise. Let a_n denote the number of binary strings of length n that do not contain consecutive 0's. Show that a_n satisfies the recurrence $a_n = a_{n-1} + a_{n-2}$ subject to the initial conditions $a_1 = 2$, $a_2 = 3$. If b_n denotes the number of ternary strings of length n that do not contain consecutive 0's, find (and prove!) a recurrence and initial conditions for b_n.